IS FUZZY SETS A GOOD SET THEORY FOR MULTIPLE-VALUED LOGIC?

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This abstract has two parts. The first one is devoted to the representation of algebras and classes of algebras by Fuzzy Sets. The second one is devoted to some applications to algebraic logic specially to the applicability of truth table methods to propositional calculi of some logics.

1.- Classes of algebras representable by Fuzzy Sets

Let L and A be algebras of the same type and let X be a set. Let $P_L(X)$ be the algebra of L-fuzzy sets with operations pointwise defined by the operations of L. It is well known that this algebra is isomorphic to the direct product of copis of L ($P_L(X) \approx \Pi_X L$).

Definition 1.- A mapping m : A ----> L is said to be a *multiple-valuation* if m is a morphism.

Given a set M of multiple-valuations from A to L, the relation $\div$ defined on A by $a \div b$ iff$_{DEF}$ $m(a) = m(b)$ for every m of M, is a congruence relation on A. Let A/M be the quocient algebra and let [a] be the class that contains a.

Proposition 1.- A/M is representable as subalgebra of $P_L(M)$.
(The mapping f: A/M ---> $P_L(M)$ defined by $f([a])(m) = m(a)$ is a monomorfism)

Definition 2.- A family of multiple-valuations M from A to L is said to be *separating* if $A/M \approx A$.

Hence, proposition 1 gives a representation theorem for A if M is a separating family. The reciprocal of this result is also true as the following proposition shows.

Proposition 2.- If an algebra A is representable as subalgebra of $P_L(X)$ for some algebra L and some set X, the set X can be identified as a separating family of multiple-valuations.

(If f:A--->$P_L(X)$ is a monomorfism we can identify every $x \in X$ with the mapping m_x:A--->L defined by $m_x(a) = f(a)(m)$).

Finally if Σ is an equational class of algebras, the following theorem holds:

Theorem 1.- There exists an algebra L of Σ such that every algebra A of Σ is representable as subalgebra of $P_L(X)$ for some set X if, and only if, there exists an algebra L of Σ that contains all subdirectly irreducible algebras of Σ as subalgebras.

An algebra L of Σ satisfying the condition of theorem 1, is called an *algebra of values* for Σ and such a class Σ is said to be *representable by Fuzzy Sets* .

In this sense, Hilbert algebras, Implication algebras, relative pseudo-complemented lattices, contrapositionally complemented lattices, semi-complemented lattices, pseudo-complemented lattices, Heyting algebras and topological boolean algebras are not representable by Fuzzy Sets. But Tetravalent modal algebras, Lukasiewicz and Post algebras of order n, Ockham algebras and some subvarieties of this (varietiies $H_{p,q}$, varieties of p-symmetric algebras, De Morgan algebras, Kleene algebras and Boolean algebras) are representables by Fuzzy Sets.*

2.- Some applications to logic.

The problem of algebraization of logic began with Tarski's earliest papers. At the begining, the most part of works in algebraic logic are devoted to the study of the so called Tarski-Lindembaun algebra. Posteriorly some works began to associate a class of algebras to a logic. In the book "An algebraic Aproach to Non-Classical Logic", Rasiowa defines Logics as classes of the Standard Systems of implicative extensional propositional calculi, she had defined. This classes are defined by means of the class of algebras associated to every Standard System and all this classes of algebras are equationally definible. In this frame, we point out the equivalence between the possibility to stablish a calculi by means of truth tables in a logic and the possibility to represent the class of algebras associated to this logic, by Fuzzy Sets . This logical calculi is an efective proof calculi if the algebra of values of the associated class of algebras, is finite. For example, the n-valued logic of lukasiewicz has the Post algebras as its associated class of algebras and this class has the Post algebra defined on a chain of n elements, as algebra of values. Therefore an efective proof calculi based on truth tables valued on the n-elements Post algebra exists, as Rasiowa pointed out.

REFERENCES.-

Rasiowa, H. “An algebraic approach to non-classical logic" North-Holland (1974)
Balbes,R & Dwinger,P. “Distributive lattices" University of Misouri Press (1974)
Font,J.M. & Rius,M “A four-valued modal logic arising from Monteiros last algebras" Math. Preprints n.73 Fac.Math. Univ. Barcelona 1989
Garcia,P & Esteva,F “Representation theorem for Ockham algebras and their subdirectly irreducible algebras" To be published in Studia Logica.

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* See references for nomenclatura, definitions and basic results of this classes of algebras.