# ON FUZZY PREFERENCE MODELING : A SURVEY

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**Abstract**

An approach to the axiomatics of preference modelling based on results obtained by Ovchinnikov and Roubens and by Fodor is suggested. In the framework of this approach, general definitions of strict preference $P$, indifference $I$ and incomparability $J$ are established. It is shown that in some cases $P$ is based on the Luckasiewicz $t$-norm and $I, J$ are defined with min $t$-norm.

Let $R$ be a valued preference relation on $A : \{a, b, c, \ldots\}$, i.e. a mapping $A^2 \to [0,1]$.

We introduce a strict preference relation $P$, an indifference relation $I$ and an incomparability relation $J$ associated with $R$ as valued binary relations defined by

$$P(a,b) = p\,[R(a,b), R(b,a)] \tag{1}$$
$$I(a,b) = i\,[R(a,b), R(b,a)] \tag{2}$$
$$J(a,b) = j\,[R(a,b), R(b,a)] \tag{3}$$

where $p, i; j$ are functions

$$p, i, j : [0,1]^2 \to [0,1]$$

such that
$p$ is nondecreasing in its first argument and nonincreasing in its second argument,
$i$ is nondecreasing with respect to both arguments,
$j$ is nonincreasing with respect to both arguments.

Introducing the inverse of $R, R^{-1}$, the complementary of $R, R^c$ and the dual $R, R^d = (R^{-1})^c$, the following formal relations are considered :

$$R = P \cup I \tag{4}$$
$$R^d = P \cup J \tag{5}$$

If we use the $t$-conorm $S$ on a model for union connective, $T$ as the $t$-norm associated to $S$ and the negation function $n$ (i.e. a strictly decreasing and continuous function $[0,1] \to [0,1]$ which satisfies limit conditions : $n(0) = 1$, $n(1) = 0$) as a model for the complementary, we obtain from (1)-(5) if $x = R(a,b)$, $y = R(b,a)$

$$x = S\,[p(x,y), i(x,y)] \tag{6}$$
$$ny = S\,[p(x,y), j(x,y)] \tag{7}$$

with

$$S(x,y) = n^{-1}T\left[nx, ny\right]. \tag{8}$$

Fodor has proved in [2] that $T$ satisfying (6)-(8) has zero divisors ($T(x,nx) = 0$) which implies – see [8] – when $T$ is continuous that $T(x,y) = \emptyset^{-1}W(\emptyset x, \emptyset y)$ where $\emptyset$ is an automorphism from $[0,1]$ to $[0,1]$ and $W$ is the Luckasiewicz $t$-norm and

$$\begin{gathered} T(x,ny) \leq p(x,y) \leq \min(x,ny) \\ T(x,y) \leq i(x,y) \leq \min(x,y) \\ T(nx,ny) \leq j(x,y) \leq \min(nx,ny). \end{gathered}$$

This result is linked to one theorem of Alsina [1] who proved that the equation

$$x = S_2\left[T_1(x,Ny), T_2(x,y)\right]$$

when $N$ is an involutive strict negation ($NNx = x$) and $T_1, T_2$ are $t$-norms and $S_2(x,y) = NT_2(Nx,Ny)$, has a unique parametrized family of solutions

$$\begin{aligned} Nx &= \emptyset^{-1}\left[1 - \emptyset x\right] \\ T_1(x,y) &= \emptyset^{-1}\pi\left[\emptyset x, \emptyset y\right] \\ T_2(x,y) &= \emptyset^{-1}W\left[\emptyset x, \emptyset y\right] \end{aligned}$$

where $\pi$ is the product $t$-norm.

Let us now consider some particular cases :

(i) $p(x,y) = T(x,y)$, $i(x,y) = T(x,y)$;

Alsina proved in [1] that equation (6) has no solution.

(ii) $p(x,y) = \min(x,y)$, $i(x,y) = T(x,y)$ with $T(x,nx) = 0$, $j(x,y) = T(nx,ny)$;

Equations (6) et (7) are fulfiled and $p$ satisfies $T$-antisymmetry ($T\left[p(x,y), p(y,x)\right] = 0$).

(iii) $p(x,y) = T(x,y) = \emptyset^{-1}W(\emptyset x, \emptyset y)$, $i(x,y) = \min(x,y)$, $j(x,y) = \min(nx,ny)$, $nx = \emptyset^{-1}\left[1 - \emptyset x\right]$;

Ovchinnikov and Roubens have proved in [9] the following result : If $p(x,y) = g(x,Ny)$ with positiveness response ($g$ is nondecreasing with respect to both arguments) and antisymmetry ($\min\left[P(a,b), P(b,a)\right] = 0$ or in an equivalent way $g(x,Nx) = 0$), $i(x,y) = h(x,y)$, $j(x,y) = h(Nx,Ny)$ with positiveness response and symmetry, the system of equations

$$\begin{cases} x = S\left[g(x,Ny), h(x,y)\right] & (R = P \cup I) \\ 0 = T\left[g(x,Ny), h(x,y)\right] & (P \cap I = \emptyset) \\ 0 = T\left[g(x,Ny), h(Nx,Ny)\right] & (P \cap J = \emptyset) \\ 0 = T\left[h(x,y), h(Nx,Ny)\right] & (I \cap J = \emptyset) \end{cases}$$

has the parametrized family of solutions

$$\begin{cases} g(x, Ny) = \emptyset^{-1} \max(0, \emptyset x - \emptyset y) = \emptyset^{-1} W(\emptyset x, \emptyset N y) \\ h(x, y) = \min(x, y) \\ Nx = \emptyset^{-1} [1 - \emptyset x] \\ T(x, y) = \emptyset^{-1} \max(0, \emptyset x + \emptyset y - 1) = \emptyset^{-1} W(\emptyset x, \emptyset y) \\ S(x, y) = \emptyset^{-1} \min(1, \emptyset x + \emptyset y) \end{cases}$$

(iv) $p(x, y) = T(x, ny)$, $i(x, y) = \min(x, y)$, $j(x, y) = \min(nx, ny)$.

Fodor has proved in [2] the following result :

Consider system (1)-(15) when $p$ satisfies min-antisymmetry. The *unique* parametrized family of solutions is given by $T$ is a continuous Archimedean $t$-norm with zero divisors $nx = T^{\rightarrow}(x, 0)$ where $T^{\rightarrow}$ is the residuation of $T$.

$$p(x, y) = T[x, ny], \; i(x, y) = \min(x, y), \; j(x, y) = \min(nx, ny)$$

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