MANY - VALUED LOGIC AND THE TREATMENT OF FUZZY RELATIONS AND OF GENERALIZED SET EQUATIONS

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1. Background and notation

The basic idea to connect the fields of fuzzy sets and many-valued logic is very simple and almost presents itself: the generalized membership degrees $\mu_A(a)$ for elements a (of a fixed universe of discourse $\underline{X}$) with respect to a fuzzy subset A of $\underline{X}$ are to be taken as generalized truth values of a generalized membership predicate ε. Therefore I use to write "$a \varepsilon A$" instead of "$\mu_A(a)$" and usually identify a formula like "$a \varepsilon A$" with its truth value.

Suppose that a t-norm t is given in advance which at least is left continuous and which is the semantical counterpart of a conjunction connective $\&_t$ in many-valued logic. We suppose the quantifiers $\forall, \exists$ to mean inf, sup and take 1 as the only (positively) designated generalized truth value. Hence we write $\models H$ to indicate that the well-formed formula H (always) has truth value one. With $\&_t$ we connect an implication connective $\rightarrow_t$ having the residuation

$$seq_t(x,y) =_{def} \sup\{z \mid t(x,z) \leq y\}$$

as truth function and a negation connective $-_t$ defined as

$$-_t H =_{def} H \rightarrow_t \underline{f}$$

with $\underline{f}$ as propositional constant for the truth value zero. We use class terms $\{x \| H(x)\}$ to denote that fuzzy subset A of the universe of discourse $\underline{X}$ for which $\mu_A(x)$ always is the truth value of H(x), i.e. for which the truth values of "$x \varepsilon A$" and "H(x)" always coincide. And we use the fuzzified inclusion relation:

$$A \subseteq_t B =_{def} \forall x (x \varepsilon A \rightarrow_t x \varepsilon B) .$$

2. Fuzzy relations and their properties

We are interested in properties and operations peculiar to fuzzy (binary) relations. A standard result, e.g., is the monoton-

icity property

$$\text{if } \models R \subseteq_t S, \text{ then } \models R \circ_t T \subseteq_t S \circ_t T \qquad (\mathbf{X})$$

of the commonly defined relational product

$$R \circ_t T =_{def} \{ (a,b) \mid \exists x ((a,x) \varepsilon R \ \&_t \ (x,b) \varepsilon T) \} .$$

The interesting point is that for the approach with many-valued logic it is quite natural to conjecture (and prove) this monotonicity more general as

$$\models R \subseteq_t S \longrightarrow_t R \circ_t T \subseteq_t S \circ_t T ,$$

which, for the truth values, is an inequality instead of simply connecting two cases with fixed truth values as in (**X**).

Another point is how to define properties of fuzzy relations. Usual definitions are e.g.

$$R \text{ transitive } =_{def} \models \forall x \forall y \forall z (xRy \ \&_t \ yRz \longrightarrow_t xRz) ,$$
$$R \text{ symmetric } =_{def} \models \forall x \forall y (xRy \longrightarrow_t yRx) ,$$
$$R \text{ asymmetric } =_{def} \models \forall x \forall y (xRy \longrightarrow_t -_t yRx) .$$

Here, as usual, we write xRy instead of $(x,y) \varepsilon R$. It is routine to give results like

$$R \text{ transitive iff } \models R \circ_t R \subseteq_t R$$

or to discuss special classes of fuzzy relations like similarity relations (i.e. equivalence relations), ordering relations etc. Results concerning such fuzzy relations are numerous and well known. But, there are natural generalizations: properties of fuzzy relations themselves may be fuzzy, i.e. may hold true for fuzzy relations only to some degree. As examples consider

$$\text{Trans}(R) =_{def} \forall x \forall y \forall z (xRy \ \&_t \ yRz \longrightarrow_t xRz) ,$$
$$\text{Asymm}(R) =_{def} \forall x \forall y (xRy \longrightarrow_t -_t yRx) ,$$
$$\text{Irrefl}(R) =_{def} \forall x (-_t xRx) .$$

Now it will be possible in a canonical way to define fuzzy relations which are similarity relations, ordering relations, ... <u>to some degree</u>. Well known result can be generalized in a natural way. Remember e.g. that in the crisp case

if R transitive and R irreflexive, then R asymmetric holds true. Indeed, now in full generality one can prove

$$\models \text{Trans}(R) \ \&_t \ \text{Irrefl}(R) \longrightarrow_t \text{Asymm}(R) .$$

Hence, the degrees of transitivity and irreflexivity of a fuzzy relation R determine a lower bound for the degree of asymmetry of R .

3. Fuzzy relation equations and their solvability

Some engineering approaches which use fuzzy sets to connect fuzzy values A, B of fuzzy variables u, v via a fuzzy relation R are using the (full) image of a set under a relation: $B = R''A$ defined, in the present notation, as

$$R''A =_{def} \{ y \mid \exists x (x \varepsilon A \ \&_t \ (x,y) \varepsilon R) \}.$$

Very simple kinds of expert systems, so called fuzzy controllers, start e.g. from some basic facts formulated as a finite list

$$B_i = R''A_i \ , \quad i = 1,\dots,n \qquad (+)$$

of "control rules" with given fuzzy data A_i, B_i $(i = 1,\dots,n)$. The engineers problem is to determine R out of (+). If one treats this as the mathematical problem to solve a system of (set) equations soon it becomes obvious that system (+) has a solution only under very restrictive conditions. With many-valued logic and a generalized, i.e. fuzzy identity

$$A \equiv B =_{def} A \subseteq_t B \ \&_t \ B \subseteq_t A$$

one can take a more general attitude and consider the formula

$$\exists R (\bigwedge_{i=1}^{n} R''A_i \equiv B_i) \qquad (\S)$$

of many-valued logic which (using $\bigwedge$ as finite iteration of the conjunction $\&_t$) formulates that "system (+) has a solution".

It is possible to determine resp. to estimate the truth value of (§) only using the given "data" A_i, B_i. Additionally, the ŁUKASIEWICZ (i.e. "one minus") negation of $\equiv$ defines a distance function in usual sense. Thus the truth value of (§) can be used to get information about "best possible" approximate solutions of (+). This can be a starting point to discuss the "quality" of an engineering model (+) of a given process.

The present way to approach the solvability of (systems of) fuzzy relation equations is, yet, not only suitable for that type of equations, relation equations with the relational product $R \circ_t S$ instead of $R''A$ and also fuzzy-arithmetical equations with $a + b$ instead of $R''A$ (for fuzzy "numbers" a, b in the sense of fuzzified intervals) can be treated in the same manner. Further extensions seem to be possible.

4. Some open problems

With regard to fuzzy properties of fuzzy relations till now there are only a few, scattered results. A (kind of) coherent treatment is lacking.

For fuzzy relation equations one may be interested (i) to apply this approach to wider classes of equations, (ii) to give a more detailled discussion of the idea that the ŁUKASIEWICZ negation of the generalized identity "measures" approximation quality, (iii) to use that idea to determine suitable t-norm conjunctions via suitable distance functions for fuzzy sets, and (iv) to discuss links between the structure of the set equations and the structure of their sets of solutions (or also their fuzzy sets of "approximate" solutions ?).

For the generalized identity used here a problem is if its definition is (really) well suited or if other definitions would do the job better. And a (deep ?) problem is to understand its relationship with the use of generalized identities e.g. in the D. SCOTT's approach toward HEYTING-valued sets.

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