SIMILARITY RELATIONS
AND THE SEMANTICS OF FUZZY LOGIC

Extended Abstract

Enrique H. Ruspini
Artificial Intelligence Center
SRI International
Menlo Park, California, U.S.A.

INTRODUCTION

In this brief communication,[1] we summarize the results of recent research on the conceptual foundations of fuzzy logic [4]. This research resulted in the formulation of several semantic models that interpret the major concepts and structures of fuzzy logic in terms of the more primitive notion of resemblance and similarity between "possible worlds," i.e., the possible states, situations or behaviors of a real-world system. The metric structures representing this notion of proximity are generalizations of the accessibility relation of modal logics [1].

Possibilistic reasoning methods may be characterized, by means of our interpretation, as approaches to the description of the relations of proximity that hold between possible system states that are logically consistent with existing evidence, and other situations, which are used as reference landmarks. By contrast, probabilistic methods seek to quantify, by means of measures of set extension, the proportion of the set of possible worlds where a proposition is true.

Our discussion will focus primarily on the principal characteristics of a model, discussed in detail in a recent technical note [2], that quantifies resemblance between possible worlds by means of a similarity function that assigns a number between 0 and 1 to every pair of possible worlds. Introduction of such a function permits to interpret the major constructs and methods of fuzzy logic: conditional and unconditional possibility and necessity distributions and the generalized modus ponens of Zadeh on the basis of related metric relationships between subsets of possible worlds.

THE APPROXIMATE REASONING PROBLEM

Our semantic model of fuzzy logic is based on two major conceptual structures: the notion of possible world, which is the basis for our unified view of the approximate reasoning problem [3], and a metric structure that quantifies similarity between pairs of possible worlds.

If a reasoning problem is thought of as being concerned with the determination of the truth-value of a set of propositions that describe different aspects of the behavior of a system, then a *possible world* is simply a function (called a valuation) that assigns a unique truth value to every proposition in that set and that, in addition, is consistent with the rules of propositional logic. The set of all such possible worlds is called the *universe of discourse*.

In any reasoning problem, knowledge about the characteristics of the class of systems being studied combined with observations about the particular system under consideration restricts the extent of possible worlds that must be considered to a subset of the universe of discourse, called the *evidential set*, which will be denoted $\mathcal{E}$.

The purpose of the inferential procedures utilized in any reasoning problem may be characterized as that of establishing if, for a given proposition p, either $\mathcal{E} \Rightarrow p$ or $\mathcal{E} \Rightarrow \neg p$. In approximate reasoning problems, as illustrated in Figure 1, such determination is, by definition, impossible: there are some possible worlds in the the evidential set where the hypothesis is true and some where it is false.
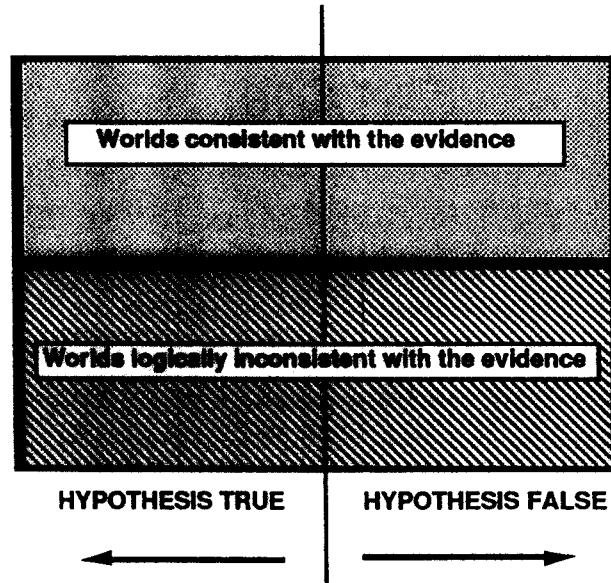

Figure 1: The approximate reasoning problem

SIMILARITY FUNCTIONS AND GENERALIZED IMPLICATION

In the view of fuzzy logic proposed by our model the purpose of possibilistic methods is the description of the evidential set by characterization of the resemblance relations that hold between its elements and elements of other sets used as reference landmarks.

To represent similarity or resemblance between possible worlds we introduce a binary function S that assigns a value between 0 and 1 to every pair of possible worlds w and w'. Reasonable assumptions show that the relation S has the properties of a $\circledast$-transitive fuzzy equivalence relation where $\circledast$ is a continuoys triangular norm.

The correspondence between propositions and subsets of possible worlds simplifies the interpretation of the classical rule of modus ponens as a rule of derivation based on the transitive property of set inclusion. If three propositions p, q and r are such that the set of possible worlds where p is true is a subset of the set of possible worlds where q is true, and if such set is itself a subset of the set of worlds where r is true, then the modus ponens simply states that the set of p-worlds is a subset of the set of r-worlds.

The conventional relation of set inclusion, based on the binary truth-value structure of classical logic, allows only to state that a set of possible worlds is a subset of another or

that it is not. Introduction of a metric structure in the universe of discourse, however, permits the quantification of the degree by which a set is included into another. Every set of possible worlds, as illustrated in Figure 2, is a subset of some neighborhood of any
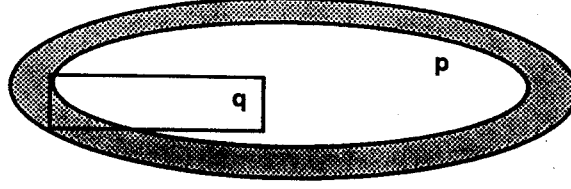

Figure 2: Degree of implication

other set. The minimal amount of "stretching" that is required to include a set of possible worlds q in a neighborhood of a set of possible worlds p, given by the expression $I(p\,|\,q) = \inf_{w' \vdash q} \sup_{w \vdash p} S(w, w')$, is called the *degree of implication*.

The degree of implication function has the important transitive property expressed by $I(p\,|\,q) \geq I(p\,|\,r) \circledast I(r\,|\,q)$, which is the basis of the generalized modus ponens of Zadeh. As illustrated in Figure 3, this important rule of derivation tells us how much the set of p-worlds should be stretched to encompass q on the basis of knowledge of the sizes of the neighborhoods of p that includes r and of r that includes q.
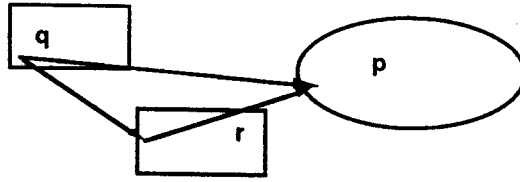

Figure 3: The generalized modus ponens

A notion dual to the degree of implication is that of *degree of consistence*, which quantifies the amount by which a set must be stretched to intersect another, and that is given by the expression $C(p\,|\,q) = \sup_{w' \vdash q} \sup_{w \vdash p} S(w, w')$.

POSSIBILISTIC DISTRIBUTIONS

Although the transitive property of the degree of implication essentially provides the bases for the conceptual validity of the generalized modus ponens, this rule of derivation is typically expressed by means of necessity and possibility distributions.

An *unconditioned necessity distribution* given the evidence $\mathcal{E}$ is any function defined over propositions that bounds by below the degree of implication function, i.e., any function satisfying the inequality $\mathrm{Nec}(p) \leq I(p\,|\,\mathcal{E})$. Correspondingly, an *unconditioned possibility distribution* is any upper bound for the degree of consistence function, i.e., $\mathrm{Poss}(p) \geq C(p\,|\,\mathcal{E})$.

The definition of conditional possibility and necessity distributions makes use of a form of inverse of the triangular norm denoted $\oslash$ and defined by the expression

$$a \oslash b = \sup\{\, c : \; b \circledast c \leq a \,\}.$$

Using this function, it is possible to define conditional possibilistic distributions as follows:

Definition: A function $\text{Nec}(\cdot|\cdot)$ is called a *conditional necessity distribution* for $\mathcal{E}$ if

$$\text{Nec}(q|p) \leq \inf_{w \vdash \mathcal{E}} \left[I(q\,|\,w) \oslash I(p\,|\,w) \right].,$$

Definition: A function $\text{Poss}(\cdot|\cdot)$ is called a *conditional possibility distribution* for $\mathcal{E}$ if

$$\text{Poss}(q|p) \geq \sup_{w \vdash \mathcal{E}} \left[I(q\,|\,w) \oslash I(p\,|\,w) \right].$$

GENERALIZED MODUS PONENS

The *compositional rule of inference* or generalized modus ponens of of Zadeh is a generalization of the corresponding classical rule of inference that may be used even when known facts do not match the antecedent of a conditional rule. The interpretation provided by our model explains the generalized modus ponens as an extrapolation procedure that uses knowledge of the similarity between the evidence and a set of possible worlds p (the antecedent proposition), and of the proximity of p-worlds to q-worlds, to bound the similarity the latter to the evidential set. The actual statement of the generalized modus ponens for necessity distributions in terms of similarity structures makes use of a family $\mathcal{P}$ of satisfiable propositions that partitions the universe of discourse:

Theorem *(Generalized Modus Ponens for Possibility Functions)*: Let $\mathcal{P}$ be a partition and let q be a proposition. If $\text{Poss}(p)$ and $\text{Poss}(q|p)$ are real values, defined for every proposition p in $\mathcal{P}$, such that

$$\text{Poss}(p) \geq C(p\,|\,\mathcal{E}), \quad \text{Poss}(q|p) \geq \sup_{w \vdash \mathcal{E}} \left[I(q\,|\,w) \oslash I(p\,|\,w) \right],$$

then the following inequality is valid:

$$\sup_{\mathcal{P}} \left[\text{Poss}(q|p) \circledast \text{Poss}(p) \right] \geq C(q\,|\,\mathcal{E}).$$

A dual result holds for necessity functions.

REFERENCES

[1] G.E. Hughes and M.J. Creswell. *An Introduction to Modal Logic*. New York: Methuen, 1972.

[2] E.H. Ruspini. *On the Semantics of Fuzzy Logic*. Technical Note No. 475, Artificial Intelligence Center, SRI International, Menlo Park, California, 1989.

[3] E.H. Ruspini, *Approximate Reasoning: Past, Present, Future*. Information Sciences, forthcoming, 1990.

[4] L.A. Zadeh. A theory of approximate reasoning. In D. Michie and L.I. Mikulich, editors, *Machine Intelligence 9*, New York: Halstead Press, 149–194, 1979.