FUZZY RELATIONS FROM A CATEGORICAL POINT OF VIEW

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Let L be a complete Heyting algebra - i.e. a complete lattice $(L,\leq)$ satisfying the infinite distributive law

$$\alpha \wedge \left(\bigvee_{i \in I} \beta_i \right) \quad = \quad \bigvee_{i \in I} (\alpha \wedge \beta_i) \qquad .$$

Further every set X can be provided with the crisp equality (= Kronecker symbol) δ defined by $\delta(x,x) = 1$ and $\delta(x,y) = 0$ whenever $x \neq y$.

1. It is well known that every L-fuzzy subset μ of X (i.e. $\mu : X \longrightarrow L$) can be identified with a subobject $((S(\mu),E),m)$ of (X,δ) in the sense of Higgs's topos L-SET (cf. [2],[3],[4]) as follows

$$S(\mu) \quad = \quad \{\alpha \cdot \chi_{\{x\}} \mid x \in X \ , \ \alpha \leq \mu(x)\}$$

$$E(\alpha_1 \cdot \chi_{\{x_1\}} \ , \ \alpha_2 \cdot \chi_{\{x_2\}}) \quad = \quad \alpha_1 \wedge \alpha_2 \wedge \delta(x_1,x_2)$$

$$m(\alpha_1 \cdot \chi_{\{x\}} \ , \ y) \quad = \quad \alpha_1 \wedge \delta(x,y)$$

and vice versa.

2. Let $\mathcal{C}$ be a finitely complete category. A $\mathcal{C}$-subobject $R \xrightarrow{<a,b>} X \times X$ of $X \times X$ is an equivalence relation (cf. [1]) iff

(i) The diagonal of $X \times X$ factors through $<a,b>$. (Reflexivity)

(ii) There exists $\tau : R \longrightarrow R$ s.t. $b \cdot \tau = a$ and $a \cdot \tau = b$. (Symmetry)

(iii) If

$$\begin{array}{ccc} T & \xrightarrow{q} & R \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle b} \\ R & \xrightarrow[a]{} & X \end{array} \qquad \text{is a pullback} \quad ,$$

then $<b \cdot p, a \cdot q> : T \longrightarrow X \times X$ factors through $<a,b>$. (Transitivity)

3. <u>Theorem</u> For every L-fuzzy relation $\mu : X \times X \longrightarrow L$ the following assertions are equivalent

   (a) $((S(\mu),E),m)$ is an equivalence relation in the sense of Higgs's topos L-SET .

   (b) μ satisfies the subsequent conditions

   $$\mu(x,x) = 1 \quad , \quad \mu(x,y) = \mu(y,x) \quad , \quad \mu(x,y) \wedge \mu(y,z) \ \leq \ \mu(x,z) \quad .$$

References

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