MV-ALGEBRAS AND ULAM'S GAME WITH LIES

(Extended Abstract)

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Someone thinks of a number between one and one million (which is just less than 2^{20}*). Another person is allowed to ask up to twenty questions, to each of which the first person is supposed to answer only yes or no. Obviously the number can be guessed by asking first: Is the number in the first half million ? then again reduce the reservoir of numbers in the next question by one-half, and so on. Finally the number is obtained in less than* $\log_2(1000000)$*. Now suppose one were allowed to lie once or twice, then how many questions would one need to get the right answer ?*

S.M.ULAM, *Adventures of a Mathematician* Scribner's, New York, 1976, page 281

Ulam's game has an alternative interpretation in the framework of communication with feedback [6], [16]. Here, a low power transmitter P is sending binary numbers to a receiver Q (the Questioner). Distortion may transform into $1-b$ any bit b traveling from P to Q; however, after receiving each bit, Q can communicate with no distortion to P whether b or $1-b$ was actually received. We naturally expect that a long binary number x contains more distorted bits than a short one. Thus it is natural to consider the generalization of Ulam's game where each number x is associated with a maximum number $m(x)$ of lies, or distortions, depending on x. We give an algebraic analysis of the simplest generalizations of Ulam's game with an infinite search space. We describe a one-one correspondence between a fairly large class of Ulam games and of MV algebras. We refer to [3], [4], and [8, § 3] for background.

A *Ulam game* is a triple $G = (S, m, L)$, where S is a structure, m is a function from S to $\mathbb{N} = \{0,1,2,...\}$, and L is a set of subsets of S. S is called the *search space*, m is the *lie bound*, and L is the *language* of the game. Each $Q \in L$ is called a *question*, and is canonically identified with the question "does the unknown x belong to the set Q ?"; in particular, the empty set corresponds to the trivial question "is x outside the search space ?". Apart from the case of no lies, which is widely considered in combinatorial search theory [1], in the literature one finds examples of Ulam's games with $m \neq 0$ where S is a linearly ordered set and L is the set of initial segments, or S is a topological space and questions are measurable subsets of S; see, e.g., [13],[14]. The above quotation deals with the Ulam game $G = (S, m, L)$ with S the set of numbers between one and one million, m the function constantly equal to 2, and L the powerset of S. Optimal searching strategies are described in [5] and [12].

Games $G = (S, m, L)$ and $G' = (S', m', L')$ are *isomorphic* iff there exists an isomorphism $\theta : S \cong S'$ such that $m(x) = m'(\theta(x))$ for each $x \in S$, and for every subset X of S we have that $X \in L$ iff $\{\theta(x) \mid x \in X\} \in L'$.

A state of knowledge of the Questioner (the second person in the above quotation) is completely described by a function assigning to each point $x \in S$ the quantity $q(x)$ of answers currently falsified by x, where $q(x) \in \{0,1,2,...,m(x), \text{"too many"}\}$. Upon identifying "too many" with $m(x)+1$, and dividing $q(x)$ by $m(x)+1$, we can represent the current state of knowledge by the relative number of answers falsified by

x, or, dually, by the quantity $d(x) = 1 - q(x)/(m(x)+1)$. By definition, a *state of knowledge* of the game $G = (S, m, L)$ is a function $d{:}S \to \mathbf{Q}$ such that for each $x \in S$, $d(x) \in \{0, 1/(m(x)+1), \ldots, m(x)/(m(x)+1), 1\}$. The quantity $d(x)$ is the *relative distance* (in units of $m(x)+1$) of x from the condition of falsifying too many answers. The *initial* state of knowledge is the function constantly equal to one (no x falsifies any answer). After questions $Q_1,\ldots,Q_t$ have been answered, the state of knowledge d is uniquely determined by his answers, as follows: For any subset Q of S, the *positive answer* $Q^{\text{yes}}: S \to \mathbf{Q}$ is defined by $Q^{\text{yes}}(x) = 1$ whenever $x \in Q$, and $Q^{\text{yes}}(x) = 1 - 1/(m(x)+1)$, otherwise. This accounts for the fact that if $x \in Q$, the number of answers falsified by x is unchanged, while if $x \notin Q$, x takes a step towards the condition of falsifying too many answers. Also define $Q^{\text{no}} = \overline{Q}^{\text{yes}}$, where $\overline{Q} = S \setminus Q$ is the *opposite* question. Then the state of knowledge after questions $Q_1,\ldots,Q_t$ have respectively been answered $e_1,\ldots, e_t$, (where $e_j \in \{\text{yes, no}\}$ for each $j = 1,\ldots,t$) is the function $Q_1^{e_1} \bullet\ldots\bullet Q_t^{e_t}$, where $\bullet$ is *Lukasiewicz conjunction*, $a \bullet b = \max(0, a + b - 1)$. A state of knowledge d is *final* iff $d(x) \neq 0$ holds for exactly one element x in the search space.

Given a Ulam game $G = (S, m, L)$ *the MV algebra A_G associated to G* is the MV algebra of rational valued functions on S, with pointwise operations, generated by the answers to the questions of L.

Bounded numbers of lies A Ulam game $G = (S, m, L)$ is *finite* iff S is a finite set—or, a finite discrete topological space—and L is the powerset of S.

THEOREM. *The map $G \to A_G$ induces a one-one correspondence between isomorphism classes of finite Ulam games, and isomorphism classes of finite MV algebras.*

COROLLARY. *Isomorphism classes of finite Ulam games are in one-one correspondence with isomorphism classes of finite dimensional C*-algebras.*

For each $n = 2,3,4,\ldots$, we let I_n be the Lukasiewicz chain with n elements, i.e., the MV algebra $\{0, 1/(n-1), 2/(n-1), \ldots, (n-2)/(n-1), 1\}$ with natural MV operations. Let X be a Boolean (i.e., a totally disconnected, compact, Hausdorff) space. We denote by $C(X, I_n)$ the MV algebra of all continuous functions from X into I_n, the latter being equipped with the discrete topology. In the light of Epstein's representation theorem [7, Theorem 16], we say that an MV algebra B is a *Post* MV *algebra of order n* iff $B \cong C(X, I_n)$ for some Boolean space X. Thus in particular, Post MV algebras of order 2 are the same as Boolean algebras.

Generalizing the notion of finite game, we say that a Ulam game $G = (S, m, L)$ is *Boolean* iff S is a Boolean space, m is a continuous function from S to $\mathbf{N}$, the latter being equipped with the discrete topology, and L is the set of all clopen subspaces of S. We say that G has *constant lie bound* iff, in addition, m is a constant function, say $m(x) = k$ for all $x \in S$.

THEOREM. *For each $k = 0, 1,\ldots,$ the map $G \to A_G$ induces a one-one correspondence between isomorphism classes of Boolean Ulam games with constant lie bound equal to k, and isomorphism classes of Post MV algebras of order $k+2$.*

COROLLARY. *Under the above map, finite Ulam games with constant lie bound correspond to finite Post MV algebras of finite order.*

COROLLARY. *The map $G \to A_G$ induces a one-one correspondence between isomorphism classes of Boolean Ulam games, and isomorphism classes of finite products of Post MV algebras of finite order.*

Arbitrarily large numbers of lies Generalizing the examples of the previous section, we say that a Ulam game $G = (S, m, L)$ is *quasiboolean* iff G obeys the following three conditions:

(i) S is a set, and L is a field of subsets of S. In addition, L is *reduced,* i.e., for any two points x' and x'' in S, there is a question Q in L such that $x' \in Q$ and $x'' \notin Q$, [15, p. 18].

(ii) For each $n = 0, 1, 2, \ldots$, the set $m^{-1}(n) = \{x \in S \mid m(x) = n\}$ is a member of L.

(iii) For every ultrafilter (maximal filter) U of L which is not determined by any point of S (i.e., $\cap U = \emptyset$, [15, p. 15]) we have $\lim_{x \to U} m(x) = \infty$, in the sense that for each $r \in \mathbf{N}$ there is $Y \in U$ such that $m(y) > r$ for all $y \in Y$.

Intuitively, Condition (i) means that any two distinct points of S can be distinguished by the questions available in L. Condition (ii) enables the Questioner to ask "is x among the elements for which the rules of the game allow you to lie at most n times ?". Condition (iii) states that any "nonstandard" point x^* outside S determined by a maximal consistent set U of questions of L is actually outside the scope of any searching strategy, because the number of lies one can tell about the "standard" points near x^* tends to infinity.

THEOREM. *The map $G \to A_G$ induces a one-one correspondence between isomorphism classes of quasiboolean Ulam games, and isomorphism classes of Archimedean MV algebras with greatest singular element, and which are generated by their singular elements together with the constant element* 1.

COROLLARY. *The map $(H,u) \to \Gamma(H,u) \to G_{\Gamma(H,u)}$ induces a one-one correspondence between isomorphism classes of Archimedean l-groups with strong unit, with a greatest singular element, generated by their singular elements and by the strong unit u, and isomorphism classes of quasiboolean Ulam games.*

Let us finally recapitulate our algebraization:

ULAM GAME	*MV-ALGEBRA*
game $G = (S, m, L)$	algebra A
search space S	maximal ideal space of A
element $x \in S$	maximal ideal J in A
lie bound for x is $m(x)$	cardinality of quotient chain A/J is $m(x)+2$
G is finite with k lies	A is a finite Post MV algebra of order $k+2$
G is finite with variable lie bound	A is a finite MV algebra
G is infinite Boolean with k lies	A is an infinite Post MV algebra of order $k+2$
G is Boolean	A = finite product of Post MV algebras of finite order
G is Boolean with no lies	A is a Boolean algebra
quasiboolean, possibly unbounded lie bound	Archim., generated by singulars, with greatest singular
initial state of knowledge	the constant element 1
arbitrary state of knowledge	arbitrary element of A
question	subset of maximal ideal space
opposite question	complementary subset
question in L	support of singular element of A
answer	dual of singular element
answer "yes, x is outside the search space"	dual $1 - s$ of greatest singular element
state of knowledge after some answers	conjunction of corresponding duals of singulars
final state of knowledge	nonzero multiple of an atom

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