APPROXIMATION IN MINIMAX ALGEBRA

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1. A view of approximation

We have a set S of objects and an object t which does not belong to S. We seek an object s in S which most resembles t.

The elements of S are generated by a spanning subset or basis, B, say. Thus

$$s = \underset{j\in B}{\Gamma}(c_j(j)) \tag{1-1}$$

where Γ is a suitable associative combining operator and $\{c_j\}$ are weighting operators from some coefficient space C. We may define two maps: τ^* which takes objects to coefficient-sets, and τ which takes coefficient-sets to objects in S:

$$\tau^* : t \to \{\hat{c}_j\} \qquad \tau : \{c_j\} \to \underset{j\in B}{\Gamma}(c_j(j)) \tag{1-2}$$

The compositions $\tau^*\circ\tau$, $\tau\circ\tau^*$ act as projection operators, mapping elements of C, T respectively to approximants in $\tau^*(T)$, $\tau(C)$ respectively, and there hold:

$$\tau\circ\tau^*\circ\tau = \tau, \qquad \tau^*\circ\tau\circ\tau^* = \tau^* \tag{1-3}$$

A familiar instance of this occurs in Euclidean-space projection, but another occurs when $T = (T,\le)$, $C = (C,\le)$ are partially-ordered, with a pair of isotone maps

$$\tau : C \to T \quad \text{and} \quad \tau^* : T \to C \tag{1-4}$$

such that

$$\tau^*\circ\tau(c) \ge c \ \forall c \in C; \ \tau\circ\tau^*(t) \le t \ \forall t \in T \tag{1-5}$$

The maps τ, τ^* are known as each other's residuals and satisfy (1-3).

2. Linear minimax algebra

Consider the following algebraic structure M (known as max-algebra):

$$M = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes), \text{ where } x \oplus y = \max(x,y),\ x \otimes y = x + y \qquad (2\text{-}1)$$

An $n \times n$ real matrix $A = [a_{ij}]$ is given, and a real n-tuple $t = [t_i]$. We require to find n real numbers $x = \{x_1, \ldots, x_n\}$ such that the n-tuple $y = [y_i]$ shall be a Chebychev-best approximation to t, where y is defined by

$$y_i = \max\{a_{i1} + x_1, \ldots, a_{in} + x_n\} \quad (i = 1, \ldots, n) \qquad (2\text{-}2)$$

This problem arises, for example, in machine-scheduling. Thus, we ask for a Chebychev-best solution to a system of linear equations in max-algebra:

$$A \otimes x = t \qquad (2\text{-}3)$$

To solve problem (2-3) we introduce <u>min-algebra</u>, the system M′, dual to M:

$$M' = (\mathbb{R} \cup \{+\infty\}, \oplus', \otimes'), \text{ where } x \oplus' y = \min(x,y),\ x \otimes' y = x + y. \qquad (2\text{-}4)$$

For any $m \times n$ matrix $A = [a_{ij}]$ over M we may define the <u>conjugate</u> $n \times m$ matrix A^* over M′ by negation-and-transposition:

$$A^* = [-a_{ji}] \qquad (2\text{-}5)$$

<u>Proposition 2.1</u> <u>Under the natural partial order of n-tuples of real numbers, the maps τ, τ^* are residuals, where</u>

$$\tau : x \to A \otimes x \qquad \tau^* : t \to A^* \otimes' t \quad \blacksquare \qquad (2\text{-}6)$$

The n-tuple $z = A\otimes(A^*\otimes' t) = \tau \circ \tau^*(t)$ is the projection of t onto the column-space $A\otimes$.

Suppose the greatest componentwise difference between t and z is 2h.

<u>Proposition 2.2</u> <u>A solution to the Chebychev approximation problem (2-3) is given by</u>

$$x = h \otimes (A^* \otimes' t) \quad \blacksquare$$

3. Rational max-algebra

In max-algebra we may carry out rational operations. We define "powers of x" by

$$x^{(r)} = x \otimes x \otimes \dots \otimes x;\ x^{(0)} = 0;\ x^{(-r)} = -rx \quad (r > 0), \tag{3-1}$$

and if U, V are expressions in max-algebra, we define

$$\frac{U}{V} = U \otimes V^{(-1)} \tag{3-2}$$

The theory so arising has application to the optimisation and representation of piecewise-linear functions.

Given an N-tuple $\{a_1, \dots, a_N\}$, we may define a maxpolynomial function P by

$$P : x \to \sum_{j=1}^{N} \oplus (a_j \otimes x^{(j)}) \tag{3-3}$$

where $\sum\oplus$ stands for iterated use of the operator $\oplus$. Consider the map

$$\tau : \{a_1, \dots, a_N\} \to P \tag{3-4}$$

Proposition 3.1 The map τ forms a residuation with the map τ^*:

$$\tau^* : P \to \{a_j\} \text{ where } a_j = \min_x \,(P(x) - jx) \quad (j = 1, \dots, N) \quad \blacksquare$$

Consider the approximation of the monomial x^r $(r > 1)$ as a maxpolynomial of the form (3-3).

From Proposition 3.1, the required coefficients are given by

$$a_j = \min_x \,(x^r - jx) \tag{3-5}$$

whence

$$a_j = (1 - r)\left(\frac{j}{r}\right)^{r/r-1} \tag{3-6}$$

A polynomial is a linear combination of monomials, and the composition of monomials into polynomials can be rendered into rational processes of max-algebra. Consider the polynomial $2x^3 - x^2$. If the monomials x^3 and x^2

are approximated by the maxpolynomials P_3 and P_2 respectively, then $2x^3 - x^2$ is approximated by

$$\frac{[P_3(x)]^{(2)}}{P_2(x)} \tag{3-7}$$

Rational max-algebraic processes are computationally efficient, all relevant procedures being of merely linear complexity. The methods are especially convenient when the given functions are piecewise linear, as occurs for example in certain problems of locational analysis.

4. References

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