Book Review of

"Fuzzy Sets and Their Applications" by Vilem Novak

Adam Hilger, Bristol and Philadelphia, 1989, 248 p.
Translated from the Czech

This small book has two merits. First, it gives an account of research done in Czechoslovakia and some other Eastern Europe countries in the eighties. Second, it proposes a special view of fuzzy sets that directly stems from Goguen's logic of vague concepts (1968), further developed by Pavelka (1979) at the end of the seventies, and to which the author has himself recently contributed (Novak, 1990). As a consequence there is a strong emphasis on multiple-valued logic in the book, and this is what makes it worth reading, since it does not completely fit the usual approach to fuzzy sets, as currently expressed by Zadeh in the framework of possibility theory rather than multiple-valued logic. Noticeably indeed there is nothing about possibility theory in the book which is thus orthogonal to other recent monographs on fuzzy sets such as Klir and Folger (1988) and the one cosigned by this reviewer (Dubois and Prade, 1988).

The book is divided into two parts, corresponding to theory and applications, respectively. Part 1 starts with a useful refresher on classes and sets, relations and structures (especially lattices), first-order logic, and formal grammars and languages. Chapter two is a presentation of fuzzy set theory based on residuated lattices (first introduced by Goguen). In this view, membership grades belong to a complete infinitely distributive lattice $(L, \vee, \wedge, \mathbb{1}, \mathbb{O})$ where $\vee$ and $\wedge$ correspond to inf and sup, $\mathbb{1}$ is the top element of L, and $\mathbb{O}$ the bottom element of L. Moreover L is equiped with a multiplication $\otimes$ such that $(L, \otimes, \mathbb{1})$ is a commutative monoid, $\otimes$ is increasing in the wide sense in both places. L is also equipped with a residuum $\rightarrow$ (that will act as an implication connective), that is decreasing in the wide sense in the first place, and increasing in the second place, and that is adjoint to $\otimes$, in the following sense

$$\forall\ \alpha, \beta, \gamma \in L, \alpha \otimes \beta \leq \gamma \text{ if and only if } \alpha \leq \beta \rightarrow \gamma$$

This structure denoted $(L, \wedge, \vee, \mathbb{O}, \mathbb{1}, \otimes, \rightarrow)$ is called a residuated lattice, and serves for the generation of fuzzy set-theoretic connectives. For instance there are two fuzzy set intersections, based on $\wedge$ or $\otimes$, and complementation is based on operation $\neg\alpha = \alpha \rightarrow \mathbb{O}$. When $L = [0,1]$, this structure is the one underlying Lukasiewicz logic, and called MV-algebra by C.C. Chang (1958). The review of fuzzy sets includes extension principle, measures of fuzziness, fuzzy relations (equations, similarities and orderings) fuzzy groups, fuzzy topology, and probability of fuzzy events, basic operations on fuzzy numbers. However some of these topics are only touched upon, and recent results in fuzzy relational equations, fuzzy topology, fuzzy events, ordering on fuzzy numbers are not reported.

Chapter 3 reviews Zadeh's approach to the semantics of natural language and linguistic variables. It is strongly based on Zadeh's first paper on that topic (Zadeh, 1971), a paper that is very often forgotten by many authors and that is fully accounted for here. This chapter closes with an interesting discussion on linguistic approximation, which especially presents the approach by Esragh and Mamdani (1979).

Chapter 4 is called fuzzy logic, and is probably one of the most interesting of the book. It contains an extensive presentation of an unusual many-valued first order logic based on residuated lattices that extends Pavelka's propositional many-valued logic. This logic includes a set of logical constants in the language, and the interpretations of these constants form a residuated lattice. These constants enable fuzzy sets of (non-logical) axioms to be defined and fuzzy sets of formulas to be produced by inference rules. Important theorems about completeness are proved, that can be found nowhere else in any previous book on fuzzy sets. The following result due to Pavelka is especially worth noticing (theorem 4.14, p. 142) : It is not possible to build a complete multiple-valued logic of that kind on [0,1] using a discontinuous implication operation. This chapter also contains a section on linguistic truth-values, and one on the generalized modus ponens. However these topics are much less developed than the multiple-valued logic, which seems to be an extension of Lukasiewicz logic, although the author does not refer to axiomatizations of this logic.

The second part is a survey of some applications of fuzzy sets that were carried out in the past especially by the author and other Czech or Eastern Germany researchers. It is interesting insofar as it gives access to new entries in the fuzzy set literature that were not known by the Western community. Topics surveyed are decision analysis, fuzzy controller, man-machine communication, fuzzy systems modeling, fuzzy system behaviour evaluation, fuzzy automata, fuzzy algorithm and fuzzy programming. Again these chapters are more interesting for the information they provide about work done beyond the (late) iron curtain, rather than for the originality of insight they would give on the considered topics. However, many of the presented techniques have been used in real world applications, in Czechoslovakia or other neighbouring countries.

The book ends by a discussion of "general problems of fuzzy set theory" that recalls some methods for the determination of the membership function, the links between the notions of vagueness and indiscernibility, the links between fuzzy sets and Vopenka's semi-set theory (Vopenka, 1979), and a criticism of the approach to fuzzy sets based on triangular norms. The book also includes a short glossary of fuzzy sets with translation of the basic vocabulary into Russian, French, German and Czech.

The challenging thesis of this book is that only Lukasiewicz implication ($\alpha \to \beta = \min(1, 1 - \alpha + \beta)$) leads to completeness of fuzzy logic based on the unit interval. The universality of this claim can be debated. Indeed, the residuated lattice structure on which the author builds his theory is only one possible approach to fuzzy sets. Other types of structures are worth considering as well, for instance De Morgan algebras based on triangular norms, as studied by the Barcelona team (Alsina et al., 1983). It all depends upon the choice of the basic connectives. In the case of the residuated lattice, the infinum and supremum of the lattice serve as conjunction and disjunction, and an implication symbol is defined independently ; then negation derives from the implication, and the only way to recover Zadeh's fuzzy complementation is to adopt Lukasiewicz implication (since then $\alpha \to 0 = 1 - \alpha$), or equivalently to use as a multiplication operation the "bold" intersection $\alpha \otimes \beta = \max(0, \alpha + \beta - 1)$. On the contrary in De Morgan algebras, the primitive connectives are conjunction (a triangular norm) and negation (Trillas strong negations, which are involutive). Implication is then induced from conjunction and negation ($\alpha \Rightarrow \beta = \neg(\alpha \wedge \neg\beta)$) or from conjunction through residuation (but the identity $\alpha \Rightarrow 0 = \neg\alpha$ is then given up). Hence the claims pertaining to Lukasiewicz implication as the only one leading to completeness of the logic seem to make sense so far as one restricts oneself to logical axioms that are compatible with the residuated lattice ($[0,1]$, min, max, 1, 0, $\max(0, \alpha + \beta - 1)$, $\min(1, 1 - \alpha + \beta)$).

Related to this debate is the definition of connectives that fit the lattice. It is an extension of the trivial requirement that if $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ then any connective f should be such that $f(\alpha_1,\beta_1) = f(\alpha_2,\beta_2)$. In the multiple-valued version, equality is replaced by soft equivalence $\leftrightarrow$ ($\alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha)$) and f fits the residuated lattice if and only if (see p. 39) there exists natural integers k_1, k_2 such that

$$(\alpha_1 \leftrightarrow \beta_1)^{k_1} \otimes (\alpha_2 \leftrightarrow \beta_2)^{k_2} \leq f(\alpha_1,\alpha_2) \leftrightarrow f(\beta_1,\beta_2) \qquad (1)$$

where the powers to k_1 and k_2 are in the sense of $\otimes$. The author indicates that discontinuous operations, such as Gödel implication

$$\begin{aligned} \alpha \Rightarrow \beta &= 1 \text{ if } \alpha \leq \beta \\ &= \beta \text{ otherwise} \end{aligned}$$

do not fit the residuated lattice and should be there by rejected. However this implication is obtained by residuation from min($\alpha \Rightarrow \beta = \sup\{\gamma \mid \min(\alpha,\gamma) \leq \beta\}$) and it is then possible to consider a residuated lattice structure where $\otimes = \wedge$. The residuum is Gödel implication $\Rightarrow$. Considering the corresponding equivalence $\alpha \Leftrightarrow \beta = 1$ if $\alpha = \beta$ and $\alpha \wedge \beta$ otherwise, Gödel implication then fits the residuated lattice in the sense that

$$(\alpha_1 \Leftrightarrow \beta_1) \wedge (\alpha_2 \Leftrightarrow \beta_2) \leq (\alpha_1 \Rightarrow \beta_1) \Leftrightarrow (\alpha_2 \Rightarrow \beta_2) \qquad (2)$$

Of course viewing ([0,1], $\wedge$, $\vee$, $\wedge$, $\Rightarrow$) as a residuated lattice, negation does not any longer correspond to Zadeh's fuzzy set complementation ($\alpha \Rightarrow 0 \neq 1 - \alpha$) but becomes intuitionistic (e.g. Dubois and Prade, 1984). Intuitionistic fuzzy set theory is absent from the book, although investigated some time ago (Takeuti and Titani, 1984). Anyway, even if the choice of the residuated lattice structure made by the author is debatable, the idea expressed by inequalities (1) and (2) is quite important and deserves further investigation, in the scope of other systems of multiple-valued logic.

On the whole this book raises interesting questions in multiple-valued logic as a basis for fuzzy set theory, and should be recommended to any logician and any mathematician of Artificial Intelligence. Mathematicians of social sciences should also be interested, provided that they consider fuzzy set theory as a useful tool in their field.

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