Are grey sets novel ?
Grey sets are nothing but interval-valued fuzzy sets

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There has been a considerable amount of literature produced by Chinese researchers about so-called grey sets, grey numbers, grey systems since the mid-eighties, following an idea advocated by Prof. Deng Julong [3]. Some of these works were published in BUSEFAL (see issues n° 36, 38, 39, 40, 41, 42, 44) and there exists a publication specially devoted to this topic [2]. Usually they are presented as a fairly new, powerful tool, making reference almost only to the grey set and system literature. Especially the link with fuzzy sets is not always really acknowledged, even in a recently published paper devoted to the relationship between grey numbers and fuzzy numbers (and where fuzzy numbers are rather presented as a particular case) [22]. The aim of this short note is to clarify this point.

A grey set G on a referential X is defined by two functions $\underline{\mu}_G$ and $\bar{\mu}_G$ from X to [0,1] such that $\underline{\mu}_G \leq \bar{\mu}_G$, called "lower and upper subordinate functions" [21][22]. Clearly to each $x \in X$ is attached an interval $[\underline{\mu}_G(x), \bar{\mu}_G(x)] \subseteq [0,1]$. $\underline{\mu}_G(x)$ and $\bar{\mu}_G(x)$ are called "the lower and upper subordinate degrees of x relative to G". This interval may be understood as containing (the) possible values of the degree of membership of x to G. We have recognized a Φ-fuzzy set, first considered by Sambuc [19], i.e. an interval-valued fuzzy set. This is a well-known particular case of type 2-fuzzy set [12], a notion first introduced by Zadeh [23] (a type 2-fuzzy set is a set whose membership degrees are fuzzy sets of [0,1], or more generally of a set L of valuations). See also [5] pp. 30-31 and pp. 63-66.

Φ-fuzzy sets can be encountered in the fuzzy set literature with slightly different interpretations. Let us especially mention

— a sub-specified fuzzy set, whose membership degrees, at least for some elements, are ill-known. This sub-specification may result of the appreciation of experts or of the mathematical constraint(s) defining the fuzzy set ;

— a "flou" set $\mathcal{G}$ [10] which is a pair of two ordinary subsets $(\underline{G}, \bar{G})$ of X such that $\underline{G} \subseteq \bar{G}$ where $\underline{G}$ is understood as a set of typical elements while $\bar{G} - \underline{G}$ is interpreted as a set of elements considered as peripheral for $\mathcal{G}$;

— a sub-definite set $\mathcal{G}$ [14] which is a 4-tuple $(G_+, G_-, \bar{m}, \underline{m})$ where G^+ (resp. G^-) is the set of elements which are known to belong (resp. not to belong) to $\mathcal{G}$ and $\bar{m} \leq |X - G_-|$ and $\underline{m} \geq |G_+|$ are respectively an upper and a lower bound of the number of elements which belong to $\mathcal{G}$;

— a twofold fuzzy set G [6][7], is defined by a pair $(\underline{\mu}_G, \bar{\mu}_G)$ such that $\{x \in X, \underline{\mu}_G(x) > 0\} \subseteq \{x \in X, \bar{\mu}_G(x) = 1\}$, where $\underline{\mu}_G$ (resp. $\bar{\mu}_G$) is the membership function of the set of elements which belong more or less certainly (resp. possibly) to G (in the sense of possibility theory [24]) ;

— an intuitionistic fuzzy set [1] defined by a pair (μ_G, ϑ_G) where $\mu_G(x)$ is a degree of membership and $\vartheta_G(x)$ is a degree of non-membership for x, both degrees belonging to [0,1], with the constraint $\forall x, \mu_G(x) + \vartheta_G(x) \leq 1$, or equivalently $\mu_G(x) \leq 1 - \vartheta_G(x)$, thus the pair $(\mu_G, 1 - \vartheta_G)$ is equivalent to a pair of lower and upper membership functions $(\underline{\mu}_G, \bar{\mu}_G)$;

— a rough set [16][17] and more generally a rough fuzzy set [8] is a (fuzzy) set G defined on X whose membership function on a *coarsened* referential X/R (where X/R is a quotient space which represents a partition of X induced by an equivalence relation) is bounded from below and from above respectively by

$$\forall\, t \in X/R,\ \underline{\mu}_G(t) = \inf\{\mu_G(x), x \in t\}$$
$$\bar{\mu}_G(t) = \sup\{\mu_G(x), x \in t\}$$

More generally, a fuzzy rough (fuzzy) set [8][13] is defined when considering a similarity relation or a fuzzy partition instead of R or X/R respectively.

Particular cases of grey sets are ordinary sets and fuzzy sets obtained when $\underline{\mu}_G = \bar{\mu}_G$, as well as the so-called Deng grey sets [22] where $\bar{\mu}_G$ is the characteristic function of an ordinary subset S of X and $\underline{\mu}_G = 0$ everywhere. It may be interpreted as follows, values outside S do not belong to the grey set definitely while values in S *may* belong to it. Another kind of grey set which has a particular meaning from a possibility distribution specification point of view ([25][20][9]), obeys to the following constraints, where S is an ordinary subset of X

$$\forall\, x \notin S,\ \underline{\mu}_G(x) = 0 \quad \text{and} \quad \bar{\mu}_G(x) = 1$$
$$\forall\, x \in S,\ \bar{\mu}_G(x) = 1$$

while $\underline{\mu}_G$ remains unconstrained over S. In terms of possibility distribution it means that the values x in S are *at least* possible at the degree $\underline{\mu}_G(x)$ and that nothing is known outside of S. Dually, a grey set such that $\forall\, x \notin S$, $\underline{\mu}_G(x) = 0$ and $\bar{\mu}_G(x) = 1$ and $\forall\, x \in S$, $\underline{\mu}_G(x) = 0$ corresponds to the knowledge that the value $x \in S$ is *at most* possible to the degree $\bar{\mu}_G(x)$.

Type 2-fuzzy sets have been studied from a theoretical point of view (e.g. [11]), arithmetic operations on type 2-fuzzy numbers have been defined and studied [4][5], as well as type 2-fuzzy relations [27]. Interval-valued fuzzy sets have been advocated by Zadeh himself under the name of "ultra fuzzy set" (e.g. [26]). Let us also mention the use of interval-valued fuzzy sets which has been proposed in decision analysis and in economical studies long time ago (e.g. [15][18]), among many other works. So it would be a pity that researchers in grey systems ignore results available in the fuzzy set literature. Certainly interval-valued fuzzy sets are of a considerable interest in general, however a clear interpretation of grey sets and a clarified relationship with other existing proposals are necessary from a scientific point of view.

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