INITAL INQUISITION OF GREY GROUP
(INITAL INQUISITION OF COMPLEX FUZZY GROUP)

YANG ZHIMIN, LIU HUTAO

Handan Teacher's College, Handan, Hebei, China

ABSTRACT: In this paper the definition and theorems of grey subgroup are introduced. And on this basis, the relationship between grey subgroup and fuzzy and general subgroup is discussed. There are also studies of grey normal subgroup.

KEYWORDS: Gery subsemigroup, grey submonoid, grey subgroup and grey normal subgroup.

I. INTRODUCTION

A · Rosenfeld defined fuzzy group in 1971. Wu Heqin and Wang Qingyin gave the concepts and properties of grey set in 1988. We shall study the grey group on this basis.

Definition 1: Let G be a non-vacuous set. If G satisfies:

(1) G is closed under product, or $\forall x,y\in G\rightarrow xy\in G$.

(2) The product in G satisfies law of association, or $\forall x,y,z\in G\rightarrow (xy)z=x(yz)$.

(3) There exists a unit in G, or $\exists e\in G$, s. t. $\forall x\in G\rightarrow xe=ex=x$.

(4) Every element in G are invertible, or $\forall x\in G$, $\exists x^{-1}\in G$, s. t. $xx^{-1}=x^{-1}x=e$.

Then G is called a group.

If G satisfies (1), (2) and (3), then G is called a monoid. If G satisfies (1) and (2), then G is called a semigroup.

Definiion 2: Let G be a semigroup, A be a non-vacuous subset of G. If $\bar{\mu}_A(xy)\geqslant\min\{\bar{\mu}_A(x),\bar{\mu}_A(y)\}$, $\underline{\mu}_A(xy)\geqslant\min\{\underline{\mu}_A(x),\underline{\mu}_A(y)\}$, $\forall x,y\in G$, then A is called a grey subsemigroup of G.

If $\underline{\mu}_A(xy)\geqslant\min\{\bar{\mu}_A(x),\bar{\mu}_A(y)\}$, $\forall x,y\in G$, then A is called a strong grey subsemigroup of G.

It's obvious, a strong grey subsemigroup is a grey subsemigroup.

Definition 3: Let G be a monoid, A be a grey subsemigroup of G. If $\bar{\mu}_A(e) \geqslant \bar{\mu}_A(x)$, $\underline{\mu}_A(e) \geqslant \underline{\mu}_A(x)$, $\forall x \in G$, then A is called a grey submonoid of G.

Definition 4: Let A be a strong grey subsemigroup of the monoid G, if $\underline{\mu}_A(e) \geqslant \bar{\mu}_A(x)$, $\forall x \in G$, then A is called a strong grey submonoid of G.

It's obvious, a strong grey submonoid is a grey submonoid.

Theorem 1: Let G be a monoid, A be a non-vacuous grey subset of G, $Aa, = \{x \mid x \in G, \underline{\mu}_A(x) \geqslant a\}(a \in [0,1])$ be a non-vacuous subset of G. If A is the strong grey submonoid of G, then for any $a \in [0,1]$, Aa is the submonoid of G.

Proof: (1) Let A be the strong grey submonoid of G and $x, y \in Aa$, then $\underline{\mu}_A(x) \geqslant a$, $\underline{\mu}_A(y) \geqslant a$. Hence $\underline{\mu}_A(xy) \geqslant \min\{\bar{\mu}_A(x), \bar{\mu}_A(y)\} \geqslant \min\{\underline{\mu}_A(x), \underline{\mu}_A(y)\} \geqslant a$, so $xy \in Aa$.

(2) Since G is the monoid, hence for any $x, y \in G$, $(xy)z = x(yz)$. So for any $x, y \in Aa$, $(xy)z = x(yz)$.

(3) Since A is the a strong grey submonoid of G, hence for any $x \in Aa$, $\underline{\mu}_A(e) \geqslant \bar{\mu}_A(x) \geqslant \underline{\mu}_A(x) \geqslant a$. So $e \in Aa$.

From (1), (2) and (3) we have Aa is the submonoid of G.

II. CONCEPTION OF GREY SUBGROUP

Definition 5: Let G be a group, A be a non-vacuous grey subset. If (1) $\bar{\mu}_A(xy) \geqslant \min\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}$, $\underline{\mu}_A(xy) \geqslant \min\{\underline{\mu}_A(x), \underline{\mu}_A(y)\}$, $\forall x, y \in G$.

(2) $\bar{\mu}_A(x^{-1}) \geqslant \bar{\mu}_A(x)$, $\underline{\mu}_A(x^{-1}) \geqslant \underline{\mu}_A(x)$, $\forall x, y \in G$.

Then A is called a grey subgroup of G.

Definiin 6: Let A be a grey subgroup of the group G, if for any $x, y \in G$, $\bar{\mu}_A(xy) = \bar{\mu}_A(yx)$, $\underline{\mu}_A(xy) = \underline{\mu}_A(yx)$, then A is called a commutative grey subgroup.

Definitin 7: Let A be a non-vacuous subset of the group G. If (1) $\underline{\mu}_A(xy) \geqslant \min\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}$, $\forall x, y \in G$.

(2) $\underline{\mu}_A(x) \geqslant \bar{\mu}_A(x)$, $\forall x \in G$.

Then A is called a strong grey subgroup.

It's obvious, a strong grey subgroup is a grey subgroup.

Theorem 2: Let G be a group, A be a gery subgroup of G, then

(1) $\bar{\mu}_A(x^{-1}) = \bar{\mu}_A(x)$, $\underline{\mu}_A(x^{-1}) = \underline{\mu}_A(x)$, $\forall x \in G$.

(2) $\bar{\mu}_A(e) \geqslant \bar{\mu}_A(x)$, $\underline{\mu}_A(e) \geqslant \underline{\mu}_A(x)$, $\forall x, y \in G$.

Proof: (1) Since A is the grey subgroup of G, then

$\bar{\mu}_A(x^{-1}) \geqslant \bar{\mu}_A(x)$, $\forall x \in G$. Also $\bar{\mu}_A(x) = \bar{\mu}_A((x^{-1})^{-1}) \geqslant \bar{\mu}_A(x^{-1})$, $\forall x \in G$, so $\bar{\mu}_A(x^{-1}) = \bar{\mu}_A(x)$, $\forall x \in G$.

In the same way we can also have $\underline{\mu}_A(x^{-1}) = \underline{\mu}_A(x)$, $\forall x \in G$.

(2) $\bar{\mu}_A(e) = \bar{\mu}_A(xx^{-1}) \geqslant \min\{\bar{\mu}_A(x), \bar{\mu}_A(x^{-1})\} = \bar{\mu}_A(x)$, $\forall x \in G$.

In the same way we can also have $\underline{\mu}_A(e) \geqslant \underline{\mu}_A(x)$, $\forall x \in G$.

Corollary: Let G be a group, A be a strong grey subgroup, then $\underline{\mu}_A(e) \geqslant \bar{\mu}_A(x)$, $\forall x \in G$.

Theorem 3: Let G be a group, A be a non-vacuous grey subset of G and Let $A_{a,b} = \{x \mid x \in G, \bar{\mu}_A(x) \geqslant a, \underline{\mu}_A(x) \geqslant b\}$. Then A is the grey subgroup of G if and only if for any $a, b \in [0, 1]$, non-vacuous subset $A_{a,b}$ is the subgroup of G.

Proof: We have known for any $a, b \in [0, 1]$, non-vacuous subset $A_{a,b}$ is the subgroup of G. Choose any $x, y \in G$, Let $\min\{\bar{\mu}_A(x), \bar{\mu}_A(y)\} = a_1$, $\min\{\underline{\mu}_A(x), \underline{\mu}_A(y)\} = b_1$. Then A_{a_1,b_1} is a subgroup of G and $x, y \in A_{a_1,b_1}$, hence $xy \in A_{a_1,b_1}$. So

$\bar{\mu}_A(xy) \geqslant a_1 = \min\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}$, $\underline{\mu}_A(xy) \geqslant b_1 = \min\{\underline{\mu}_A(x), \underline{\mu}_A(y)\}$.

Choose $x \in G$, Let $\bar{\mu}_A(x) = a_2$, $\underline{\mu}_A(x) = b_2$, then A_{a_2,b_2} is a subgroup of G and $x \in A_{a_2,b_2}$. Hence $x^{-1} \in A_{a_2,b_2}$. So

$$\bar{\mu}_A(x^{-1}) \geqslant a_2 = \bar{\mu}_A(x), \quad \underline{\mu}_A(x^{-1}) \geqslant b_2 = \underline{\mu}_A(x).$$

From definition 5 we have A is the grey subgroup of G.

Also, we known A is the grey subgroup of G. Choose any $a, b \in [0, 1]$ and Let $A_{a,b} \neq \emptyset$ and choose any $x, y \in A_{a,b}$. From theorem 2 we have $\bar{\mu}_A(y^{-1}) = \bar{\mu}_A(y) \geqslant a$, $\underline{\mu}_A(y^{-1}) = \underline{\mu}_A(y) \geqslant b$, hence $\bar{\mu}_A(xy^{-1}) \geqslant \min\{\bar{\mu}_A(x), \bar{\mu}_A(y^{-1})\} \geqslant a$, $\underline{\mu}_A(xy^{-1}) \geqslant \min\{\underline{\mu}_A(x), \underline{\mu}_A(y^{-1})\} \geqslant b$, thus $xy^{-1} \in A_{a,b}$.

So $A_{a,b}$ is the subgroup of G.

III. RELATION BETWEEN GREY SUBGROUP AND FUZZY AND GENERAL SUBGROUP

Let G be a group, A be a grey subgroup of G. If the upper and lower subordinate functions of A are equal, or $\bar{\mu}_A(x) = \underline{\mu}_A(x), \forall x \in G$, then change the grey subset A of G into the fuzzy subset. Hence change the grey subgroup A of G into the fuzzy subgroup.

Let G be a group, A be a fuzzy subgroup of G. If subordinate function of A, $\mu_A(x) \in \{0, 1\}, \forall x \in G$, then change the fuzzy subset A of G into the general subset. Hence change the fuzzy subgroup A of G into the general subgroup.

Hence general subgroup is a particular example of fuzzy subgroup. Fuzzy subgroup is a particular example of grey subgroup.

So grey subgroup $\Rightarrow$ fuzzy subgroup $\Rightarrow$ general subgroup.

Theorem 4; Let G be a group. If A is a strong grey subgroup of G, then A is the fuzzy subgroup of G.

Proof; Since A is the strong grey subgroup of G, then for all $x \in G$, $\underline{\mu}_A(x) = \underline{\mu}_A((x^{-1})^{-1}) \geqslant \bar{\mu}_A(x^{-1}) \geqslant \underline{\mu}_A(x^{-1}) \geqslant \bar{\mu}_A(x)$.

Also, from the definition of grey set we have $\bar{\mu}_A(x) \geqslant \underline{\mu}_A(x), \forall x \in G$. Hence $\bar{\mu}_A(x) = \underline{\mu}_A(x), \forall x \in G$.

So A is the fuzzy subgroup.

IV. GREY NORMAL SUBGROUP

Definition 8; Let A be a grey subgroup of the group G, a be a any element in G. Then grey subgroup aA and Aa of G defined by $\bar{\mu}_{aA}(x) = \bar{\mu}_A(a^{-1}x), \underline{\mu}_{aA}(x) = \underline{\mu}_A(a^{-1}x), \forall x \in G$ and $\bar{\mu}_{Aa}(x) = \bar{\mu}_A(xa^{-1}), \underline{\mu}_{Aa}(x) = \underline{\mu}_A(xa^{-1}), \forall x \in G$.

aA and Aa are called the left and right cosets of the grey subgroup A.

Definition 9; Let A be a grey subgroup of the group G. If the left and right cosets of A are equal, or $xA = Ax, \forall x \in G$, then A is called a grey normal subgroup.

Theorem 5: Let A be a grey subgroup of the group G, the following are equivalent:

(1) A is the grey normal subgroup of G.

(2) $\bar{\mu}_A(xyx^{-1}) \geqslant \bar{\mu}_A(y)$, $\underline{\mu}_A(xyx^{-1}) \geqslant \underline{\mu}_A(y)$, $\forall x \in G$.

(3) $\bar{\mu}_A(xyx^{-1}) = \bar{\mu}_A(y)$, $\underline{\mu}_A(xyx^{-1}) = \underline{\mu}_A(y)$, $\forall x \in G$.

(4) A is the commutative grey subgroup.

Proof: (1)$\longleftrightarrow$(4)

A is the grey normal subgroup of G. $\rightarrow \forall x \in G,\ xA = Ax.$

$\rightarrow \forall x, y \in G, \bar{\mu}_A(x^{-1}y) = \bar{\mu}_{xA}(y) = \bar{\mu}_{Ax}(y) = \bar{\mu}_A(yx^{-1}), \underline{\mu}_A(x^{-1}y) =$

$= \underline{\mu}_{xA}(y) = \underline{\mu}_{Ax}(y) = \underline{\mu}_A(yx^{-1}).$

$\rightarrow \forall x, y \in G, \bar{\mu}_A(xy) = \bar{\mu}_A(yx), \underline{\mu}_A(xy) = \underline{\mu}_A(yx).$

$\rightarrow$A is the commutative grey subgroup of G.

In the same way we can also prove A is the commutative grey subgroup of G$\rightarrow$A is the grey normal subgroup of G.

Hence (1)$\longleftrightarrow$(4).

(2)$\rightarrow$(3)

$\bar{\mu}_A(xyx^{-1}) \geqslant \bar{\mu}_A(y), \forall x, y \in G. \rightarrow \bar{\mu}_A((x^{-1})y(x^{-1})^{-1}) = \bar{\mu}_A(x^{-1}yx)$

$\geqslant \bar{\mu}_A(y) = \bar{\mu}_A(x^{-1}(xyx^{-1})x) \geqslant \bar{\mu}_A(xyx^{-1}) \geqslant \bar{\mu}_A(y), \forall x, y \in G.$

$\rightarrow \bar{\mu}_A(xyx^{-1}) = \bar{\mu}_A(y), \forall x, y \in G.$

In the same way we can also prove $\underline{\mu}_A(xyx^{-1}) \geqslant \underline{\mu}_A(y)$, $\forall x, y \in G \rightarrow \underline{\mu}_A(xyx^{-1}) = \underline{\mu}_A(y), \forall x, y \in G.$

(3)$\rightarrow$(4)

Let yx=y, then y=yx^{-1}. Thus $\bar{\mu}_A(y) = \bar{\mu}_A(xyx^{-1}), \underline{\mu}_A(y) = \underline{\mu}_A(xyx^{-1})$

$\rightarrow \bar{\mu}_A(yx) = \bar{\mu}_A(xy), \underline{\mu}_A(yx) = \underline{\mu}_A(xy).$

(4)$\rightarrow$(2)

$\bar{\mu}_A(xyx^{-1}) = \bar{\mu}_A(x^{-1}yx) = \bar{\mu}_A((x^{-1}x)y) \geqslant \min\{\bar{\mu}_A(x^{-1}x), \bar{\mu}_A(y)\}$

$= \min\{\bar{\mu}_A(e), \bar{\mu}_A(y)\} = \bar{\mu}_A(y), \forall x, y \in G.$

In the same way we can also have $\underline{\mu}_A(xyx^{-1}) \geqslant \underline{\mu}_A(y), \forall x, y \in G.$

Hence (2)$\longleftrightarrow$(3)$\longleftrightarrow$(4).

So (1)$\longleftrightarrow$(2)$\longleftrightarrow$(3)$\longleftrightarrow$(4).

Theorem 6: Let G be a group, A be a non-vacuous grey subset of G. Then A is the grey normal subgroup of G if and only if for any $a, b \in [0,1]$, non-vacuous subset $A_{a,b}=\{x \mid x \in G, \bar{\mu}_A(x) \geqslant a, \underline{\mu}_A(x) \geqslant b\}$ is a normal subgroup of G.

Proof, If A is the grey normal subgroup of G, then for any $a, b \in [0,1]$, $A_{a,b}$ is the subgroup of G (from theorem 3 we have it).

Now we shall prove the normality of $A_{a,b}$.

If $\forall x \in G, \forall y \in A_{a,b}$, then $\bar{\mu}_A(xyx^{-1}) \geqslant \bar{\mu}_A(y) \geqslant a$, $\underline{\mu}_A(xyx^{-1}) \geqslant \underline{\mu}_A(y) \geqslant b$, hence $xyx^{-1} \in A_{a,b}$.

So $A_{a,b}$ is the normal subgroup of G.

Now suppose the non-vacuos subset $A_{a,b}$ is the normal sub-goup for any $a, b \in [0,1]$. From theorem 3 we have A is the grey subgroup of G.

We shall prove the grey normality of A.

Let $x, y \in G$ and $\bar{\mu}_A(y)=a$, $\underline{\mu}_A(y)=b$, then $y \in A_{a,b}$.

Also $A_{a,b}$ is the normal subgroup, hence $xyx^{-1} \in A_{a,b}$. Thus for all $x, y \in G$, $\bar{\mu}_A(xyx^{-1}) \geqslant a = \bar{\mu}_A(y)$, $\underline{\mu}_A(xyx^{-1}) \geqslant b = \underline{\mu}_A(y)$.

So A is the grey normal subgroup of G.

REFERENCES

[1] Wu Heqin, Wang Qingyin, The Theory of Compound Fuzzy Sets, Initial Inquisition of the Theory of Grey Numbers, Proceeding of NAFIPS'88.

[2] Chen Yiyuan, FUZZY MATHEMATICS, Publishing House of Hua-zhong Technical College, 1988.

[3] Xie Bangjie, ABSTRACT ALGEBRA, Publishing House of Science and Technolagy of Shanghai, 1982.