THE CONTINUITY OF ELEMENTARY GREY FUNCTION

Liu Fushou   Yue Changan

Handan Prefectural Eduction College

Handan   Hebei   China

Abstract: On the base of the continuous grey function, we study the continuity of elementary grey function in this article.

Key word: Elementary grey function

In classical mathematics, we know that the elementary functions are all continuous in their domains (except that the domains is acnode). How are about the elementary grey functions?

Similar to the classical mathematics, we can give out the elementary grey functions as follows.

The functions obtained from the basic elementary real functions after extension are called basic elementary grey functions.

The functions obtained from the basic elementary grey functions after the algebraic operations and the compositions are called the elementary grey functions.

Now, we study the continuity of the elementary gray function.

Lemma. If $y=f(x)$ is a continuous real function. $[a,b]$, $[a_0,b_0]$ lie inside the domains of the function $f(t)$, then there must be a real number $\delta>0$, such that if $\max\{|a-a_0|, |b-b_0|\}\leq\delta$, then

$$\max\{|\inf_{t\in[a,b]} f(t) - \inf_{t\in[a_0,b_0]} f(t)|, |\sup_{t\in[a,b]} f(t) - \sup_{t\in[a_0,b_0]} f(t)|\}\leq\max\{2\sup_{|t-a_0|\leq|a-a_0|}|f(t) - f(a_0)|, 2\sup_{|t-b_0|\leq|b-b_0|}|f(t) - f(b_0)|\}$$

Proof. (1) If both supremum and infimum of $f(t)$ on $[a_0,b_0]$ are obtained from inside (a_0, b_0), then as $f(t)$ is a continuity, there must be a real number $\delta > 0$, such that if $\max\{|a-a_0|,|b-b_0|\}<\delta$,

then $\inf_{t\in[a,b]} f(t) = \inf_{t\in[a_0,b_0]} f(t) \quad \sup_{t\in[a,b]} f(t) = \sup_{t\in[a_0,b_0]} f(t)$

Such the inequality in lemma is true.

(2) If only one of supremum and infimum of $f(t)$ on $[a_0, b_0]$ is obtained at the endpoints of $[a_0, b_0]$. Suppose $\sup_{t\in[a_0,b_0]} f(t) = f(a_0)$

1) If $f(a_0) > f(b_0)$

Since $f(t)$ is a continuity, namely $\lim_{t\to a_0} f(t) = f(a_0)$, $\lim_{t\to b_0} f(t) = f(b_0)$ there is a number $\delta_1 > 0$, such that if $\max\{|a-a_0|,|b-b_0|\} < \delta_1$, then $\sup_{t\in[a,b]} f(t) = f(a)$. Therefore we have

$$\left|\sup_{t\in[a,b]} f(t) - \sup_{t\in[a_0,b_0]} f(t)\right| = |f(a) - f(a_0)| \leqslant 2\sup_{|t-a_0|\leqslant|a-a_0|} |f(t) - f(a_0)|.$$

2) If $f(a_0) = f(b_0)$

With the same reason that $f(t)$ is a continuity, a number $\delta_2 > 0$ exists, such that if $\max\{|a - a_0|, |b - b_0|\} < \delta_2$, then $\sup_{t\in[a,b]} f(t) = f(a)$ or $f(b)$. Now, there is

$$\left|\sup_{t\in[a,b]} f(t) - \sup_{t\in[a_0,b_0]} f(t)\right| (= |f(a) - f(a_0)| \text{ or } |f(b) - f(b_0)|)$$

$$\leqslant \max\left\{ 2\sup_{|t-a_0|\leqslant|a-a_0|} |f(t) - f(a_0)| ,\ 2\sup_{|t-b_0|\leqslant|b-b_0|} |f(t) - f(b_0)|\right\}.$$

In short, a number $\delta' > 0$ exists, such that if $\max\{|a - a_0|, |b - b_0|\} < \delta'$, then

$$\left|\sup_{t\in[a,b]} f(t) - \sup_{t\in[a_0,b_0]} f(t)\right| \leqslant \max\left\{ 2\sup_{|t-a_0|\leqslant|a-a_0|} |f(t) - f(a_0)|,\ 2\sup_{|t-b_0|\leqslant|b-b_0|} |f(t) - f(b_0)|\right\}.$$

From (1), we know that there is a number $\delta_0 > 0$, such that if $\max\{|a - a_0|, |b - b_0|\} < \delta_0$, then $\inf_{t\in[a,b]} f(t) = \inf_{t\in[a_0,b_0]} f(t)$.

From (2), we know that there is a number $\delta'>0$, such that if $\max\{|a - a_0|, |b - b_0|\} < \delta'$, then

$$\left|\sup_{t\in[a,b]} f(t) - \sup_{t\in[a_0,b_0]} f(t)\right| \leq \max\left\{ 2\sup_{|t-a_0|\leq|a-a_0|} |f(t) - f(a_0)|,\ 2\sup_{|t-b_0|\leq|b-b_0|} |f(t) - f(b_0)|\right\}.$$

Similar to the discussion of (2), we know that there exists a number $\delta''>0$, such that if $\max\{|a - a_0|, |b - b_0|\} < \delta''$, then

$$\left|\inf_{t\in[a,b]} f(t) - \inf_{t\in[a_0,b_0]} f(t)\right| \leq \max\left\{ 2\sup_{|t-a_0|\leq|a-a_0|} |f(t) - f(a_0)|,\ 2\sup_{|t-b_0|\leq|b-b_0|} |f(t) - f(b_0)|\right\}.$$

Let $\delta = \min\{\delta', \delta''\}$. If $\max\{|a - a_0|, |b - b_0|\} < \delta$, then the inequality in lemma is true. That's all.

Theorem. Assuming $y = f(t)$ is a continuous real function, we obtain a grey function after its extention, which is denoted $y = f(x)$ $x\in D$. D is the subset of classical rational grey numbers. Therefore, the grey function $y = f(x)$ $x \in D$ is a continuous grey function.

Proof. $\forall x_0 \in D$. Suppose $x_0 = [a_0, b_0]$. $\forall \varepsilon > 0$, $\forall x\in D$, since $x_0 = [a_0, b_0]$, so we suppose $x = [a,b]$ (because if $x = [a,\underset{\sim}{} b]$, then $d[x, x_0] \geq 1$).

Now, we come to the proof. There exists a number $\delta>0$, such that if $d[x,x_0] < \delta$, then $d[f(x), f(x_0)] < \varepsilon$.

In fact, since $y = f(t)$ is a continuity real function, for above $\varepsilon>0$, there exists two numbers $\delta_1>0$ and $\delta_2>0$, such that

If $|t - a_0| < \delta_1$, then $|f(t) - f(a_0)| < \varepsilon/3$;

if $|t - b_0| < \delta_2$, then $|f(t) - f(b_0)| < \varepsilon/3$.

From the lemma, we know that there exists a number $\delta_3>0$, such that if $\max\{|a - a_0|, |b - b_0|\} < \delta_3$, then

$$\max\{|\inf_{t\in[a,b]} f(t) - \inf_{t\in[a_0,b_0]} f(t)|, |\sup_{t\in[a,b]} f(t) - \sup_{t\in[a_0,b_0]} f(t)|\}$$

$$\leq \max\{ 2\sup_{|t-a_0|\leq|a-a_0|} |f(t) - f(a_0)|, 2\sup_{|t-b_0|\leq|b-b_0|} |f(t) - f(b_0)| \}.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3, 1\}$. When $d[x, x_0] = \max\{ |a - a_0|, |b - b_0|, |\inf[x] - \inf[x_0]|\} = \max\{ |a - a_0|, |b - b_0|\} < \delta$, we have

$$d[f(x), f(x_0)]$$

$$= \max\{|P[f(x)] - P[f(x_0)]|, |Q[f(x)] - Q[f(x_0)]|, |\inf[f(x)] - \inf[f(x_0)]|\}$$

$$= \max\{|P[f(x)] - p[f(x_0)]|, |Q[f(x)] - Q[f(x_0)]|\}$$

$$= \max\{|\inf_{t\in[a,b]} f(t) - \inf_{t\in[a_0,b_0]} f(t)|, |\sup_{t\in[a,b]} f(t) - \sup_{t\in[a_0,b_0]} f(t)|\}$$

$$\leq \max\{2\sup_{|t-a_0|\leq|a-a_0|} |f(t) - f(a_0)|, 2\sup_{|t-b_0|\leq|b-b_0|} |f(t) - f(b_0)|\} \leq 2\varepsilon/3 < \varepsilon$$

So far, we have proved that the grey function $y = f(x)$ $x\in D$ is a continuous grey function.

From the algebraic operations and the compositions properties of grey limit and the definitions of a continuous grey function, we know it's trivially true that the continuous grey functions after algebraic operations and compositions are continuous grey functions. Again from this theorem and the already known results of the elementary continuous real functions, we can obtain the following results:

The elementary grey functions are continuity in their domains (except that the domains is acnode) and the grey functions are continuity that obtained from the continuous diselementary real functions (such as the piecewise continuous function) after extension.

This is an important result. If f(x) is either an elementary grey function or a grey function that obtained from the continuous diselementary real functions after extension, x_0 lies inside the domains of f(x), then f(x) is continuity at point x_0 .

According to the definitions of continuity of a grey function, we obtain the following formula

$$\lim_{x\to x_0} f(x) = f(x_0) = f(\lim_{x\to x_0} x)$$

This shows that the order of "limit" and "f" can be reversed. This is helps lot to the calculating of the limit of an elementary grey function.

Example. Find $\lim_{x\to[2,3]}\left[\frac{([\underset{\sim}{-2,}2]+[2,2]x)^2}{x-[-1,1]} + 2^x \sin x\right]$

Solution. $\lim_{x\to[2,3]}\left[\frac{([\underset{\sim}{-2,}2]+[2,2]x)^2}{x-[-1,1]} + 2^x \sin x\right]$

$$= \frac{([\underset{\sim}{-2,}2]+[2,2][2,3])^2}{[2,3]-[-1,1]} + 2^{[2,3]} \sin[2,3]$$

$$= \frac{([\underset{\sim}{-2,}2]+[4,6])^2}{[1,4]} + [4,8][\sin 3, \sin 2]$$

$$= \frac{[\underset{\sim}{4,}16]}{[1,4]} + [4\sin 3, 8\sin 2]$$

$$= [\underset{\sim}{1,}16] + [4\sin 3, 8\sin 2] = [1+4\sin 3\underset{\sim}{,} 16+8\sin 2]$$

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