A characterization Of L-fuzzy primary submodules*

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Dedicated to Alireza Afzalipour the most generous patron of Kerman University

Abstract. In this note the definition of primary submodule of a L-fuzzy module is given, and some results, specially a characterization of L-fuzzy primary submodule, are proved.

Keywords: L-Fuzzy point, L-Fuzzy ideal, L-Fuzzy module, L-Fuzzy primary submodule.

1. Introduction.

In [4] and [7], the concept of fuzzy primary ideal and L-fuzzy primary ideal of a ring R are discussed, respectively. In this note the L-fuzzy (primary) submodule of a R-module M is defined and in this regard, the product of a L-fuzzy subset of R and a L-fuzzy subset of M is given. It is shown that [4, Definition 5.1] and [7, Definition 3.1] are special cases of L-fuzzy primary submodule definition given in this note.

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A necessary condition for being a L-fuzzy primary submodule of a L-fuzzy module is given and by an example it is shown that this is not sufficient. A characterization of L-fuzzy primary submodule of a module M is given.

2. Preiminaries

We fix $L=(L, \leq, \vee, \wedge)$ as a completely distributive lattice with a least element 0 and greatest element 1. We write "sup" and "inf" for "$\vee$" and "$\wedge$", respectively. If $a,b\in L$ we write $b \geq a$ iff $a\leq b$, and $a>b$ iff $a\geq b$ and $a\neq b$. For a nonempty set X, let $F(X)=\{A|A$ is a L-fuzzy subset of X$\}$. Then for $A,B\in F(X)$, we write $A\subseteq B$ iff $A(x)\leq B(x)$ for all $x\in X$. $A \supseteq B$ iff $B\subseteq A$, and $A\subset B$ iff $A\subseteq B$ and $A\neq B$. By a L-fuzzy point x_r of X; $x\in X$, $r\in L$, we mean $x_r\in F(X)$ defined by $x_r(y)=\begin{cases} r & \text{if} y=x \\ 0 & \text{if} y\neq x \end{cases}$, and we write $x_r\in X$. If $x_r\in X$ and $x_r\subseteq A\in F(X)$, then we write $x_r\in A$. If $A\subseteq X$, by $\chi_A\in F(X)$ we mean the characteristic function defined by

$$\chi_A(x)=\begin{cases} 1 & \text{if } x\in A \\ 0 & \text{if } x\notin A . \end{cases}$$

From now on R is a commutative ring with identity and M is a unitary R-module.

Definition 2.1. Let $A\in F(R)$, then A is called a L-fuzzy left (right) ideal of R iff for all $x,y\in R$,

(i) $A(x-y) \geq \inf(A(x), A(y))$

(ii) $A(xy) \geq A(y)$ $(A(xy) \geq A(x))$

A is called a L-fuzzy ideal of R iff it is both L-fuzzy left and L-fuzzy right ideal of R.

Definition 2.2. Let $\mu \in F(M)$, then μ is called a L-fuzzy left R-module of M iff for all $r \in R$, and $x, y \in M$,

(i) $\mu(x-y) \geq \inf(\mu(x), \mu(y))$

(ii) $\mu(rx) \geq \mu(x)$

(iii) $\mu(0)=1$

We let $I(R)$ $(I_l(R))$ and $S(M)$ be the set of all L-fuzzy (left) ideals of R and the set of all L-fuzzy left R-module of M, respectively.

Lemma 2.3. Let M=R, then $\mu \in S(M)$ iff $\mu \in I_l(R)$ and $\mu(0)=1$.

Definition 2.4. Let $I \in F(R)$ and $B \in F(M)$. Define the composition and product $IoB, IB \in F(M)$, respectively as follows: For all $w \in M$.

$$\text{(i)} \quad IoB(w) = \begin{cases} \sup \inf(I(r), B(x)) \\ w = rx \ ; \text{ for some } r \in R,\ x \in M \\ 0 \qquad \text{if } w \neq rx \ ; \text{ for all } r \in R,\ x \in M. \end{cases}$$

$$\text{(ii)} \quad IB(w) = \begin{cases} \sup \inf(I(r_1), \ldots, I(r_m), B(x_1), \ldots, B(x_m)) \\ w = \sum_{i=1}^{m} r_i x_i \ ; \text{ for some } m \in \mathbb{N},\ r_i \in R,\ x_i \in M \\ 0 \qquad \text{if } w \neq \sum_{i=1}^{m} r_i x_i \ ; \text{ for all } m \in \mathbb{N}, \\ \qquad r_i \in R,\ x_i \in M. \end{cases}$$

For $\mu \in F(M)$, the level subset μ_t of μ is defined as $\mu_t = \{x \in M \mid \mu(x) \geq t\}$; $t \in L$.

Theorem 2.5. Let $\mu \in F(M)$ and $\mu(0)=1$. Then $\mu \in S(M)$ iff for all $t \in L$, μ_t is a left R-module of M.

Definition 2.6. Let $P \in I(R)$ be nonconstant. P is called L-fuzzy prime ideal iff for any $A, B \in I(R)$,

$$AB \subseteq P \text{ implies either } A \subseteq P \text{ or } B \subseteq P.$$

For $P \in I(R)$ and $\mu \in S(M)$ we let $P_* = \{x \in R \mid P(x) = P(0)\}$ and $\mu_* = \{x \in M \mid \mu(x) = 1\}$.

Definition 2.7 [7, Definition 3.1]. For a nonconstant $Q \in I(R)$, then Q is called a L-fuzzy primary ideal of R iff for any $x_r, y_s \in R$; $x_r y_s \in Q$ implies either $x_r \in Q$ or $y_s^n \in Q$; for some $n \in \mathbb{N}$.

Definition 2.8 [3,Proposition 3.5]. For $A, B \in I(R)$. Then define $(A:B) \in I(R)$, by

$$(A:B)(x) = \sup\{C(x) \mid C \in I(R),\ C \circ B \subseteq A\}.$$

Definition 2.9. [7, Definition 3.5]. Let $I \in I(R)$, define $Rad(I) \in I(R)$ as follows:

$$Rad(I) = \begin{cases} \bigcap\limits_{P \geq I} P & : \text{ if there exist some L-fuzzy prime ideal } P \geq I \\ \chi_R & \text{ otherwise .} \end{cases}$$

3. L-fuzzy primary submodule

Definition 3.1. For $\mu, \nu \in S(M)$, ν is called a L-fuzzy submodule of μ iff $\nu \subset \mu$. In particular if $\mu = \chi_M$, then we say ν is a L-fuzzy submodule of M.

Definition 3.2. Let ν be a L-fuzzy submodule of μ.

Then ν is called L-fuzzy primary submodule of μ iff for any $r_t \in R$, $x_s \in M$;

$r_t x_s \in \nu$ implies $x_s \in \nu$ or $r_t^n \mu \subseteq \nu$ for some $n \in \mathbb{N}$.

Remark 3.3. The following theorem shows that Definition 3.2 is a suitable one for L-fuzzy primary submodule.

Theorem 3.4. If M=R, then $\nu \in F(R)$ is a L-fuzzy primary submodule of M iff ν is a L-fuzzy primary ideal of R.

Remark 3.5. Theorem 3.4 and Definition 2.8 show that Definition 3.2 is a generalization of Definition 2.7 and [4, Definition 5.1].

Theorem 3.6. Let ν be a L-fuzzy primary submodule of μ. If $\nu_t \neq \mu_t$; $t \in L$, then ν_t is a primary submodule of μ_t.

Remark 3.7. The converse of Theorem 3.6 is not true as the following example shows.

Example 3.8. Let L=[0,1], $M=R=\mathbb{Z}$. Define $\mu, \nu \in S(M)$ as follows:

$$\mu(x)=\begin{cases} 1 & \text{if } x\in 4\mathbb{Z} \\ 1/2 & \text{if } x\in 2\mathbb{Z}-4\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}, \quad \nu(x)=\begin{cases} 1 & \text{if } x=0 \\ 1/2 & \text{if } x\in 4\mathbb{Z}-\{0\} \\ 0 & \text{otherwise.} \end{cases}$$

By some manipulation we can see that for all $t \in (0,1]$, ν_t is a primary submodule of μ_t. But ν is not a L-fuzzy primary submodule of μ, because if m=5, n=4, t=1/3, s=2/3; then $m_t n_s \in \nu$, but $n_s \notin \nu$ and $m_t^k \mu \subseteq \nu$ for all $k \in \mathbb{N}$.

Corollary 3.9. Let ν be a L-fuzzy primary submodule of μ, and $\nu_* \neq \mu_*$. Then ν_* is a primary submodule of μ_* .

Corollary 3.10. Let ν be a L-fuzzy primary submodule of M. Then ν_* is a primary submodule of M.

Theorem 3.11. (a) Let N be a primary submodule of M, and α a prime element of L. Then the L-fuzzy subset $\nu \in F(M)$ defined by

$$\nu(x)=\begin{cases} 1 & \text{if } x\in N \\ \alpha & \text{if } x\notin N \end{cases} \qquad (1)$$

is a L-fuzzy primary submodule of M.

(b) Conversely any L-fuzzy primary submodule ν of M can be obtained as in (1)

By some manipulation we can see that for all $t\in (0,1]$, ν_t is a primary submodule of μ_t. But ν is not a L-fuzzy primary submodule of μ, because if m=5, n=4, t=1/3, s=2/3; then m n $\in\nu$ $I^k\rho$ $\sigma\upsilon\pi$ X ρ $|C o \mu \subseteq \nu$, $C\in I(R)\}$,then $I=(\nu:\mu)$.

Remark 3.15. Let M=R and $\mu,\nu\in S(M)$, so $\mu,\nu\in I(R)$. Then, by Theorem 3.14, $(\nu:\mu)$ reduces to Definition 2.8. Hence Definition 2.8 is a special case of Definition 3.12.

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