REMARK ON THE RIEMANN - STIELTIES INTEGRAL

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The purpose of this paper is to introduce the concept of an integral of a function f with respect to a function g both defined on a closed interval $\langle a,b\rangle$ with values from an ordered metric spaces A. This is an extension in two direction: It generalizes Riemann-Stielties integral of bounded real functions to functions with values from A. On the other hand the interval defined by Riečan [2] is obtained from our definition as a special case by taking $g(x)=x$ for all $x \in \langle a,b\rangle$. Moreover, if $A=L(R)$, where $L(R)$ is a special set of so-called fuzzy numbers, and $g(x)=x$ for all $x \in \langle a,b\rangle$ than the integral of Matłoka is obtained [1].

Let us assume that A is a partially ordered set with the following properties:

- A is a boundedly complete lattice, i.e. every non empty upper (lower) bounded subset B of A has the supremum (infimum,
- there is given a commutative and associative operation + on A with a neutral element 0, preserving the ordering,
- there is given a multiplication of elements of A by real

numbers, associative, preserving the ordering and $1\cdot a=a$ and $0\cdot a=0$, $k\cdot(a+b)=k\cdot a + k\cdot b$.

Let us additionally assume that (A,d) is a metric space satysfying the following identities:

- $d(ka,kb)=k\cdot d(a,b)$,
- $d(a+b,c+d) \leq d(a,c) + d(b,d)$,
- $d(a,b) \leq d(a+c,b+c)$.

Definition. If $f : \langle a,b\rangle \to A$ and $g : \langle a,b\rangle \to A$ are the bounded functions, P is any partition of $\langle a,b\rangle$ with points

$$P : a = x_0 \leq x_1 \leq \dots \leq x_{n-1} \leq x_n = b$$

then we put

$$L_i = \inf \{ f(x) : x \in \langle x_{i-1}, x_i\rangle \},$$

$$U_i = \sup \{ f(x) : x \in \langle x_{i-1}, x_i\rangle \},$$

$$L(f,g,P) = \sum_{i=1}^{n} L_i \cdot d(g(x_{i-1}),g(x_i)),$$

$$U(f,g,P) = \sum_{i=1}^{n} U_i \cdot d(g(x_{i-1}),g(x_i)),$$

$$(L)\int_a^b fdg = \sup \{ L(f,g,P) : P \text{ is a partition of } \langle a,b\rangle \},$$

$$(U)\int_a^b fdg = \inf \{ U(f,g,P) : P \text{ is a partition of } \langle a,b\rangle \}.$$

The function f is integrable with respect to g over $\langle a,b\rangle$ if

$$(L)\int_a^b fdg = (U)\int_a^b fdg.$$

The common value will be denoted by $\int_a^b fdg$.

Proposition 1. A bounded function f is integrable with respect to g on $\langle a,b\rangle$ iff corresponding to each $\varepsilon > 0$ there is a partition P with the property $d(U(f,g,P),L(f,g,P)) < \varepsilon$.

Proof. Assume that corresponding to each $\varepsilon > 0$ such a P exists. Then, since

$$d\left((U)\int_a^b fdg, (L)\int_a^b fdg\right) \leq d(U(f,g,P), L(f,g,P)) < \varepsilon ,$$

it follows that $(U)\int_a^b fdg = (L)\int_a^b fdg$, and f is integrable with respect to g on $\langle a,b\rangle$. Conversely assume that f is integrable with respect to g on $\langle a,b\rangle$. Then $(U)\int_a^b fdg = (L)\int_a^b fdg = \int_a^b fdg$. Therefore $\int_a^b fdg$ is the least upper bound of the sums $\{L(,f,g,P)\}$ and is the greatest lower bound of the sums $\{U(f,g,P)\}$. Hence corresponding to each $\varepsilon > 0$ there is a P_1 such that

$$d\left(\int_a^b fdg, L(f,g,P_1)\right) < \varepsilon/2$$

and a P_2 such that

$$d\left(U(f,g,P_2), \int_a^b fdg\right) < \varepsilon/2.$$

So,

$$d(U(f,g,P_2), L(f,g,P_1)) \leq d\left(U(f,g,P_2), \int_a^b fdg\right) + $$
$$+ d\left(\int_a^b fdg, L(f,g,P_1)\right) < \varepsilon .$$

Let $P = P_1 \cup P_2$, P is a refinement of both P_1 and P_2 . Therefore

$$d(U(f,g,P), L(f,g,P)) \leq d(U(f,g,P_2), L(f,g,P_1)) < \varepsilon .$$

This completes the proof.

Proposition 2. If f_1 and f_2 are integrable functions with respect to g and k, l are real numbers, then $k\cdot f_1 + l\cdot f_2$ is integrable with respect to g, too, and

$$\int_a^b (k\cdot f_1 + l\cdot f_2)dg = k\cdot\int_a^b f_1 dg + l\cdot\int_a^b f_2 dg.$$

If $f_1 \leq f_2$ then $\int_a^b f_1 dg \leq \int_a^b f_2 \, dg$.

Proof. It is straightforward.

Proposition 3. If f is integrable with respect to g on $\langle a,b\rangle$ and $c \in (a,b)$, then f is integrable with respect to g on $\langle a,c\rangle$ and $\langle c,b\rangle$ and

$$\int_a^b fdg = \int_a^c fdg + \int_c^b fdg.$$

Proof. It follows from the inequalities

$$(U)\int_a^b fdg \geq (U)\int_a^c fdg + (U)\int_c^b fdg \geq (L)\int_a^c fdg + (L)\int_c^b fdg \geq$$

$$\geq (L)\int_a^b fdg.$$

REFERENCES

[1] Matłoka,M.: On an integral of fuzzy mappings,BUSEFAL 31, 1987,45-55.

[2] Riečan,B.: Remark on an integral of M.Matłoka, Math.Slovaca 38, 1988,341-344.