THE REPRESENTATION THEOREM OF FUZZY MEASURE SPACES

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In this paper, we propose the concept of regular fuzzy measurable spaces and investigate the relation between the regular fuzzy measurable spaces and the classical measurable spaces. The representation theorem of regular fuzzy measure spaces is proved.

Keywords: σ-additive F-class, F-measure, regular F-measure space, kernel space.

1. Introduction and Preliminaries

In the paper[3], Butnariu introduced the concept of additive measure of fuzzy sets (called F-measure for short in this paper), which is somewhat different from those studied by Zadeh[1], Sugeno[8], Klement [9], Ralescu and Adams[7]. The aim of this paper is to propose the concept of regular F-measurable spaces and demonstrate the representation theorem of F-measure on the regular F-measurable spaces.

Throughout this paper, X will denote a non-empty set and B_0 the σ-algebra of Borel sets of [0, 1]. Let $F(X) = \{f: X \rightarrow [0, 1]\}$, and $P(X) = \{I_E: E \subseteq X\}$, where I_E is the indicator function of E. The elements of F(X) are called fuzzy sets in X and denoted by A,B,C,... . The operations of union, intersection, inclusion and complement of fuzzy sets are adopted in the sense of Zadeh[2] and, that of sum $\oplus$ and difference $\ominus$ of fuzzy sets are adopted in the sense of Butnariu[3].

Definition 1.1[3]. Let $\mathcal{F}$ be a class of fuzzy sets. If $\mathcal{F}$ satisfies the following conditions:

(1) $I_X \in \mathbf{F}$

(2) $A, B \in \mathbf{F} \Longrightarrow A \ominus B \in \mathbf{F}$

(3) $\{A_n, n \in N\} \subseteq \mathbf{F} \Longrightarrow \oplus_{n \in N} A_n \in \mathbf{F}$

Then $\mathbf{F}$ is called a σ-additive F-class, and $(X, \mathbf{F})$ is called a F-measurable space.

Definition 1.2[3]. Let $(X, \mathbf{F})$ be a F-measurable space and m be a non-negative function on $\mathbf{F}$ with the σ-additivity: $\{A_n, n \in N\} \subseteq \mathbf{F}$, and $\sum_{n \in N} A_n(x) \leq 1$ $(\forall x \in X) \Longrightarrow m(\oplus_{n \in N} A_n) = \sum_{n \in N} m(A_n)$. Then m is called a F-measure on $(X, \mathbf{F})$, and $(X, \mathbf{F}, m)$ is called a F-measure space (abbr. FMS).

Many properties of FMS have been investigated by Butnariu in[3], [4] and [5]. For the convenience of the further discussion in this paper, we make a note on the following remark:

Remark 1.3. Let $(X, \mathbf{F}, m)$ be a FMS. If we set $\mathbf{F}_k = \mathbf{F} \cap P(X)$, $\overline{\mathbf{F}}_k = \{E:$ $E \subseteq X$ and $I_E \in \mathbf{F}_k\}$, and define $m_k: \overline{\mathbf{F}}_k \longrightarrow [0, +\infty]$ by $m_k(E) = m(I_E)$ for each $E \in \overline{\mathbf{F}}_k$, then $\overline{\mathbf{F}}_k$ is a σ-algebra of sets and $(X, \overline{\mathbf{F}}_k, m_k)$ is a classical measure space (abbr.CMS).

Definition 1.4. Let $(X, \mathbf{F}, m)$ be a FMS. The measurable space $(X, \overline{\mathbf{F}}_k)$ will be called the kernel of the F-measurable space $(X, \mathbf{F})$, and the CMS $(X, \overline{\mathbf{F}}_k, m_k)$ will be called the kernel space of the FMS $(X, \mathbf{F}, m)$.

2. The Regular F-measurable Space and Its Representation

Definition 2.1. Let $\mathbf{F}$ be a class of fuzzy sets and $\sigma(\mathbf{F})$ the σ-algebra generated by the class of sets $\{A^{-1}(\beta): \beta \in \mathbf{B}_0, A \in \mathbf{F}\}$, i.e. $\sigma(\mathbf{F}) = \sigma(A^{-1}(\mathbf{B}_0): A \in \mathbf{F})$. We will call $\sigma(\mathbf{F})$ the σ-algebra induced by $\mathbf{F}$.

Theorem 2.2. For any $\mathbf{F} \subseteq F(X)$, $\sigma(\mathbf{F})$ is the smallest σ-algebra which makes each $A \in \mathbf{F}$ is $\sigma(\mathbf{F})$-measurable.

Proof. Straightforward.

Definition 2.3. A σ-additive F-class $\mathbf{F}$ is said to be regular if it satisfies the condition:

$$\forall \alpha \in [0, 1], \forall E \in \sigma(\mathbf{F}) \Longrightarrow \alpha . I_E \in \mathbf{F} \tag{2.1}$$

A F-measurable space $(X, \mathbf{F})$ or a FMS $(X, \mathbf{F}, m)$ is said to be regular if $\mathbf{F}$ is regular. The regular FMS is abbreviated to "RFMS".

In the following content, for any σ-algebra of sets $\mathcal{A}$, we set $F(\mathcal{A}) = \{A \in F(X): A \text{ is } \mathcal{A}\text{-measurable}\}$ and $\widetilde{\mathcal{A}} = \{I_E: E \in \mathcal{A}\}$.

Theorem 2.4. Let $\mathcal{A}$ be a σ-algebra of sets. Then $F(\mathcal{A})$ is the smallest regular σ-additive F-class containing $\widetilde{\mathcal{A}}$ and

$$\sigma(F(\mathcal{A})) = \mathcal{A} \tag{2.2}$$

We will call $(X, F(\mathcal{A}))$ the F-measurable space generated by the measurable space $(X, \mathcal{A})$.

Proof. Frist, by theorem 2.2, it is easily to prove that (2.2) holds. Next, it follows from the properties of measurable functions [6] that $F(\mathcal{A})$ is a σ-additive F-class containing $\widetilde{\mathcal{A}}$. In order to prove that $F(\mathcal{A})$ is regular, we note the fact that for each $\alpha \in [0, 1]$ and $\forall E \in \sigma(F(\mathcal{A})) = \mathcal{A}$, I_E and $\alpha . I_E$ are $\mathcal{A}$-measurable, i.e. $\alpha . I_E \in F(\mathcal{A})$. Hence $F(\mathcal{A})$ is regular. Finally, suppose $\mathbf{F}'$ is an arbitrary regular σ-additive F-class containing $\widetilde{\mathcal{A}}$, we will prove $F(\mathcal{A}) \subseteq \mathbf{F}'$. In fact, for each $A \in F(\mathcal{A})$, we have that $A = \text{limit}_n A_n$, where $A_n = \bigoplus_{i=1}^{2^n} \frac{i-1}{2^n} . I_{E_i}(n)$ and $E_i^{(n)} = \{x \in X: (i-1)/2^n < A(x) \leq i/2^n\}$. $A \in F(\mathcal{A})$ means that A is $\mathcal{A}$-measurable, so $E_i^{(n)} \in \mathcal{A}$. Since $\mathbf{F}'$ is a regular σ-additive F-class containing $\widetilde{\mathcal{A}}$, we know that $I_{E_i}(n) \in \mathbf{F}'$ and $(i-1)/2^n . I_{E_i}(n) \in \mathbf{F}'$. Consequently, $A_n \in \mathbf{F}'$ and $A = \text{limit}_n A_n \in \mathbf{F}'$. Hence $F(\mathcal{A}) \subseteq \mathbf{F}'$. The proof of theorem 2.4 is finished.

Theorem 2.5. Let $\mathbf{F}$ be a regular σ-additive $\mathbf{F}$-class. If we set $\mathbf{F}^* = F(\sigma(\mathbf{F}))$, then $\mathbf{F} = \mathbf{F}^* = F(\overline{\mathbf{F}}_k)$ and $\sigma(\mathbf{F}) = \sigma(\mathbf{F}^*) = \overline{\mathbf{F}}_k$.

Proof. It is obvious that $\mathbf{F} \subseteq \mathbf{F}^*$ and $F(\mathbf{F}_k) \subseteq \mathbf{F}$. On the other hand, we have $\widetilde{\sigma(\mathbf{F})} \subseteq \mathbf{F}$, i.e. $\mathbf{F}$ is a regular σ-additive F-class containing $\widetilde{\sigma(\mathbf{F})}$. But by theorem 2.4 we know that $\mathbf{F}^*$ is the smallest one, therefore we get $\mathbf{F} \subseteq \mathbf{F}^*$. So $\mathbf{F} = \mathbf{F}^*$. Moreover, for any $A \in \mathbf{F}$, A is $\sigma(\mathbf{F})$-measurable i.e. for all $\beta \in \mathbf{B}_0$, we have $A^{-1}(\beta) \in \sigma(\mathbf{F})$. Therefore $I_{A^{-1}(\beta)} \in \mathbf{F}^* = \mathbf{F}$. Consequently, $I_{A^{-1}(\beta)} \in \mathbf{F}_k$ and $A^{-1}(\beta) \in \overline{\mathbf{F}}_k$, i.e. A is $\overline{\mathbf{F}}_k$-measurable. This means $A \in F(\overline{\mathbf{F}}_k)$. Hence we have $\mathbf{F} \subseteq F(\overline{\mathbf{F}}_k)$ and, so $\mathbf{F} = F(\overline{\mathbf{F}}_k)$. Finally, $\sigma(\mathbf{F}) = \sigma(\mathbf{F}^*) = \overline{\mathbf{F}}_k$ follows easily from $\mathbf{F} = \mathbf{F}^* = F(\overline{\mathbf{F}}_k)$.

Theorem 2.5 shows that every F-measurable space $(X, \mathbf{F})$ can be generated by its kernel $(X, \overline{\mathbf{F}}_k)$.

3. The Representation Theorem of RFMS

Theorem 3.1. Let $(X, \mathcal{A}, u)$ be a CMS and $(X, F(\mathcal{A}))$ be the regular F-measurable space generated by $(X, \mathcal{A})$. If we define $m: F(\mathcal{A}) \longrightarrow [0, +\infty]$ by

$$m(A) = \int_X A du \qquad (\forall A \in F(\mathcal{A})) \tag{3.1}$$

Then $(X, F(\mathcal{A}), m)$ is a RFMS.

Theorem 3.2. Let $(X, \mathbf{F}, m)$ be a RFMS and $\sigma(\mathbf{F})$ be the σ-algebra induced by $\mathbf{F}$. If we define $u: \sigma(\mathbf{F}) \longrightarrow [0, +\infty]$ by

$$u(E) = m(I_E) \qquad (\forall E \in \sigma(\mathbf{F})) \tag{3.2}$$

Then $(X, \sigma(\mathbf{F}), u)$ is a CMS.

The proofs of theorem 3.1 and theorem 3.2 are straightforward.

Theorem 3.3. Every RFMS $(X, \mathbf{F}, m)$ can be represented by its kernel space $(X, \mathbf{F}_k, m_k)$ with

$$m(A) = \int_X A dm_k \qquad (\forall A \in \mathbf{F}) \tag{3.3}$$

Proof. By theorem 3.2, we nkow that $(X, \sigma(\mathbf{F}), u)$ is a CMS, where u is given by (3.2). But $\sigma(\mathbf{F}) = \overline{\mathbf{F}}_k$ from theorem 2.5 and $u(E) = m(I_E) = m_k(E)$ $(\forall E \in \overline{\mathbf{F}}_k)$, so we only need to prove (3.3). In fact, for any $\alpha \in [0, 1]$ and $E \in \sigma(\mathbf{F})$, $\alpha.I_E \in \mathbf{F}$ folloes from the regularity of $\mathbf{F}$ and $m(\alpha.I_E) = \alpha.m(I_E) = \alpha.m_k(E)$ folloes from the cotinuity of F-measure m. For any $A \in \mathbf{F}$, we have $A_n \uparrow A$, where A_n as in theorem 2.4. By the definition of the integrals with respect to m_k, we get

$$\int_X A dm_k = \text{limit}_n \int_X A_n dm_k = \text{limit}_n [\textstyle\sum_{i=1}^{2^n} \frac{i-1}{2^n}.m_k(E_i^{(n)})]$$

$$= \text{limit}_n [\textstyle\sum_{i=1}^{2^n} m(\frac{i-1}{2^n}.I_{E_i}(n))] = \text{limit}_n m(A_n) = m(A)$$

Hence (3.3) holds for all $A \in \mathbf{F}$. This ends the proof of theorem 3.3.

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