ON c – FUZZY MEASURE

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In this paper, following Sugeno's fuzzy measure, we introduce the concept of c – fuzzy measure which can be more suitable in describing subjective measure problems than fuzzy measure in some cases. Then, refering to [2], we extend the concepts of autocontinuity and null – additivity of fuzzy measure to c – fuzzy measure and obtain that autocontinuity and null – additivity of c – fuzzy measure are equivalent.

1 INTRODUCTION

The concept of measure describes the character in problems which have the property of additivity. Sugeno ([1], 1972) introduced the concepts of fuzzy measure and fuzzy integral which reduce the additivity of measure to monotonicity and continuity so as to be suitable to the nature of subjective measure. Wang ([2], 1984) introduced the concepts of autocontinuity and null – additivity of fuzzy measure and got a lot of famous results about fuzzy measures and fuzzy integrals, especially about convergence theorems.

In this paper, first, we modify fuzzy measure into c – fuzzy measure which satisfies continuity not only to monotone sequences, but also to arbitary monotone set classes whose power are not more than c (i.e. $\aleph_1$). The reason that we introduce the concept of c – fuzzy measure is that subjective measure should be continuous not only to monotone sequences, but also to arbitary monotone set classes. In addition, subjective measure problems rarely meet monotone set classes whose power are more than c. So, in definition of c – fuzzy measure, we confine the set classes to be ones whose power are not more than c. Second, similar to [2], we introduce the concepts of autocontinuity and null – additivity of c – fuzzy measure and obtain they are equivalent.

2 FUZZY MEASURE AND c – FUZZY MEASURE

Let X be a non – empty set and $\mathcal{F}$ be a σ – algebra of subsets of X.

Definition 2.1 Set function u: $\mathcal{F} \to [0, +\infty)$ is called a fuzzy measure if u satisfies

(1) $u(\emptyset) = 0$;

(2) for all A, B $\in \mathcal{F}$, $A \subset B$, there hold $u(A) \leq u(B)$;

(3) for all monotone sequences $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$, there hold $u(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} u(A_n)$. (X, $\mathcal{F}$, u) is called a fuzzy measure space.

Definition 2.2 Fuzzy measure u is called null – additive if for all E, F $\in \mathcal{F}$, $u(F) = 0$, there hold $u(E \cup F) = u(E)$.

Definition 2.3 Fuzzy measure u is called autocontinuous from above (resp. from below), if for all $A \in \mathcal{F}$, $\{B_n\}_{n=1}^{\infty} \subset \mathcal{F}$, $u(B_n) \to 0$, there hold $u(A \cup B_n) \to u(A)$ (resp. $u(A - B_n) \to u(A)$).

u is called autocontinuous if u is autocontinuous not only from above, but also from below.

Definition 2.4 A non - empty set class $\xi \subset \mathcal{F}$ is called a chain if for all $A, B \in \xi$, there hold $A \subset B$ or $B \subset A$.

Definition 2.5 σ- algebra $\mathcal{F}$ is called c - algebra, if for all chains ξ, $|\xi| \leq c$ ($|\xi|$ means the power of ξ), there hold $\bigcup \xi = \bigcup_{A \in \xi} A \in \mathcal{F}$ and $\bigcap \xi = \bigcap_{A \in \xi} A \in \mathcal{F}$.

If $\mathcal{F}$ is a c - algebra, we call $(X, \mathcal{F})$ a c - measurable space.

Definition 2.6 A fuzzy measure u on c - measurable space $(X, \mathcal{F})$ is called a c - fuzzy measure, if for all chains ξ, $|\xi| \leq c$, there hold

$$u(\bigcup \xi) = \sup u(\xi) \qquad (2.1)$$

and

$$u(\bigcap \xi) = \inf u(\xi) \qquad (2.2)$$

where $\sup u(\xi)$ stands for $\sup_{A \in \xi} u(A)$ and $\inf u(\xi)$ for $\inf_{A \in \xi} U(A)$.

Example 2.1 If $X = \{x1, x2, \ldots\}$, $\mathcal{F} = 2^X$, u is an arbitary fuzzy measure on $(X, \mathcal{F})$, then $(X, \mathcal{F})$ is a c - measurable space and u is a c - fuzzy measure on $(X, \mathcal{F})$.

Example 2.2 Let $(X, \mathcal{F}, u)$ is a fuzzy measure space. If $\mathcal{F}$ is countable, then $(X, \mathcal{F})$ is a c - measurable space and u is a c - fuzzy measure on $(X, \mathcal{F})$.

Example 2.3 Let X be an arbitary set, $\mathcal{F} = 2^X$, u is defined as

$$u(A) = \begin{cases} 1, & x_0 \in A, \\ 0, & x_0 \notin A, \end{cases}$$

then $(X, \mathcal{F})$ is a c - measurable space and u is a c - fuzzy measure on $(X, \mathcal{F})$.

3 AN EQUIVALENT DEFINITION OF c - FUZZY MEASURE

Theorem 3.1 Let $(X, \mathcal{F})$ be a c - measurable space. Fuzzy measure u is a c - fuzzy measure if and only if for all well ordered chains ξ, $|\xi| \leq c$, there hold (2.1) and (2.2).

Proof Necessity is obvious. Now, we prove sufficiency.

We only verify (2.1) is true for all chains ξ, $|\xi| \leq c$. The case to (2.2) is similar.

Let ξ be a chain, $|\xi| \leq c$. If ξ has the greatest set B (i.e. for all $A \in \xi$, it is true that $A \subset B$), it is easy to know (2.1) is true.

If ξ has not the greatest set, well order ξ to $\eta = \{A1, A2, \ldots, A_\lambda, \ldots\}_{\lambda < c}$ (where λ are ordinal numbers and c is the least one of ordinal numbers whose power are c). In this time, $\{A_\lambda\}_{\lambda < c}$ may not be a chain.

Note $Ai_1 = A1$ and $T1 = \{\lambda < c;\ \lambda > i_1, A_\lambda \in \eta, Ai_1 \subset A_\lambda\}$, then $T1 \neq \emptyset$ (if not, ξ will have the greatest set). Note i_2 is the least ordinal number of T1, $T2 = \{\lambda < c;\ \lambda > i_1, i_2, A_\lambda \in \eta, Ai_1 \subset A_\lambda, Ai_2 \subset A_\lambda\}$, then $T2 \neq \emptyset$.

Using induction principle (of general well ordered set), we can get a well ordered chain $\zeta = \{Ai_1, Ai_2, \ldots, Ai_\omega, \ldots, Ai_\lambda, \ldots\}_{\lambda < c}$.

From the supposition mentioned above, we have

$$u(\bigcup \zeta) = \sup u(\zeta).$$

It is easy to know that $\bigcup \xi = \bigcup \zeta$ and $\sup u(\xi) = \sup u(\zeta)$ and the conclusion follows.

Definition 4.1 c-fuzzy measure u on c - measurable space $(X, \mathcal{F})$ is called null - additive, if for all $A, B \in \mathcal{F}$, $u(B) = 0$, we have $u(A \cup B) = u(A)$.

DEfinition 4.2 Let $\{a_\lambda\}_{\lambda < c} \subset R^1$ be a well ordered set (to ordinal number). We note the least condensation point of $\{a_\lambda\}$ by $\underline{\lim}_{\lambda \to c} a_\lambda$ and the biggest condensation point of $\{a_\lambda\}$ by $\overline{\lim}_{\lambda \to c} a_\lambda$. We call $\lim_{\lambda \to c} a_\lambda$ exists if $\underline{\lim}_{\lambda \to c} a_\lambda = \overline{\lim}_{\lambda \to c} a_\lambda$.

Definition 4.3 c - fuzzy measure on c - measurable space $(X, \mathcal{F})$ is called autocontinuous from above (resp. from below), if for all well ordered set classes $\{A_\lambda\}_{\lambda < c} \subset \mathcal{F}$, $A \in \mathcal{F}$, $\lim_{\lambda \to c} u(A_\lambda) = 0$, there hold

$$\lim_{\lambda \to c} u(A \cup A_\lambda) = u(A)$$

(resp. $\lim_{\lambda \to c} u(A - A_\lambda) = u(A)$).

Definition 4.4 If $\{A_\lambda\}_{\lambda < c}$ is a well ordered increasing (resp. decreasing) cuain, we note $\lim_{\lambda \to c} A_\lambda = \bigcup_{\lambda < c} A_\lambda$ (resp. $\lim_{\lambda \to c} A_\lambda = \bigcap_{\lambda < c} A$).

Definition 4.5 For all well ordered set set classes $\{A_\lambda\}_{\lambda < c}$(to ordinal number λ), we note $\underline{\lim}_{\lambda \to c} A_\lambda = \bigcup_{\lambda < c} \bigcap_{\lambda \leq u < c} A_u$ and $\overline{\lim}_{\lambda \to c} A_\lambda = \bigcap_{\lambda < c} \bigcup_{\lambda \leq u < c} A_u$. If $\underline{\lim}_{\lambda \to c} A_\lambda = \overline{\lim}_{\lambda \to c} A_\lambda$, we call $\lim_{\lambda \to c} A_\lambda$ exists and define $\lim_{\lambda \to c} A_\lambda = \underline{\lim}_{\lambda \to c} A_\lambda$ or $\lim_{\lambda \to c} A_\lambda = \overline{\lim}_{\lambda \to c} A_\lambda$.

Theorem 4.1 If u is a c - fuzzy measure on measure space $(X, \mathcal{F})$, then for all well ordered set classes $\{A_\lambda\}_{\lambda < c} \subset \mathcal{F}$, we have

$$u(\underline{\lim}_{\lambda \to c} A_\lambda) \leq \underline{\lim}_{\lambda \to c} u(A_\lambda) \leq \overline{\lim}_{\lambda \to c} u(A_\lambda) \leq u(\overline{\lim}_{\lambda \to c} A_\lambda)$$

Theorem 4.2 If c - fuzzy measure u is autocontinuous from above or from below, the u is null - additive.

Theorem 4.3 Let u is a c - fuzzy measure on c - measurable space $(X, \mathcal{F})$. If u is autocontinuous from above (resp. from below) and $\{B_\lambda\}_{\lambda < c} \subset \mathcal{F}$ is a well ordered set class, $\lim_{\lambda \to c} u(B_\lambda) = 0$, then there exists $\{B_{\lambda_s}\} \subset \{B_\lambda\}$ such that $u(\overline{\lim}_{s \to c} B_{\lambda_s}) = 0$ (resp. for all $A \in \mathcal{F}$, $u(A - \overline{\lim}_{s \to c} B_{\lambda_s}) = u(A)$).

Proof From $\lim_{\lambda \to c} u(B_\lambda) = 0$, we can choice a set sequence $\{B_{\lambda_n}\}_{n < \omega} \subset \{B_\lambda\}_{\lambda < c}$ such that

$$\lim_{n \to \infty} u(B_{\lambda_n}) = 0$$

where ω is the least infinite ordinal number.

This result transfers the problem to the case of fuzzy measure and the rest of the proof can be found in [6].

Sun [6] introduced the property(S) of fuzzy measure. In the following, we extend this concept to c - fuzzy measure.

Definition 4.6 We call c - fuzzy measure u on c - measurable space $(X, \mathcal{F})$ has the property(S) if u satisfies for every well ordered set class $\{B_\lambda\}_{\lambda < c} \subset \mathcal{F}$, $\lim_{\lambda \to c} u(B_\lambda) = 0$, there exists $\{B_{\lambda_s}\} \subset \{B_\lambda\}$ such that

$$u(\overline{\lim}_{s \to c} B_{\lambda_s}) = 0$$

Theorem 4.4 c - fuzzy measure u on c - measurable space $(X, \mathcal{F})$ is autocontinuous from above if and only if u is null - additive and has property(S).

Proof Necessity can be gotten from Theorem 4.2 and 4.3. Now, we prove sufficiency.

If $A \in \mathcal{F}$, $\{B_\lambda\}_{\lambda < c} \subset \mathcal{F}$ is a well ordered set class, $\lim_{\lambda \to c} u(B_\lambda) = 0$, then there exists $\{B_{\lambda_n}\}_{n<\omega} \subset \{B_\lambda\}$ such that

$$\lim_{\lambda \to c} u(A \cup B_\lambda) = \lim_{n \to \infty} u(A \cup B_{\lambda_n})$$

This conclusion changes the proof into the case of fuzzy measure and the rest of the proof can be found in [6].

Theorem 4.6 c - fuzzy measure u on c - measurable space $(X, \mathcal{F})$ is autocontinuous from above if and only if u is autocontinuous from below.

Proof From Theorem 3.4 and 3.5, we know if c - fuzzy measure u is autocontinuous from above, then u is autocontinuous from below.

Now, we prove if u is autocontinuous from below, then u is autocontinuous from above. If the conclusion is not true, then there exists $A \in \mathcal{F}$, $\varepsilon_0 > 0$ and a well ordered set class $\{B_\lambda\}_{\lambda < c} \subset \mathcal{F}$, $\lim_{\lambda \to c} u(B_\lambda) = 0$ such that for all $\lambda < c$,

$$u(A \cup B_\lambda) > u(A) + \varepsilon_0$$

Because $\lim_{\lambda \to c} u(B_\lambda) = 0$, we can choice $\{B_{\lambda_n}\}_{n<\omega} \subset \{B_\lambda\}_{\lambda < c}$ which satisfies $\lim_{n \to \infty} u(B_{\lambda_n}) = 0$. This result make the problem to the case of fuzzy measure and the rest of the proof can be found in [6].

Theorem 4.7 c - fuzzy measure u is autocontinuous if and only if u is null - additive and has property(S).

5 EQUALITY OF AUTOCONTINUITY AND NULL - ADDITIVITY OF c - FUZZY MEASURE

In this section, we shall improve Theorem 4.7 and obtain that autocontinuity and null - additivity of c - fuzzy measure are equivalent.

First of all, we introduce the concept of atom of set class.

Definition 5.1 Let $\{A_\lambda;\ \lambda \in \Lambda\}$ be a set class of X, Λ is the index set. For every $I \subset \Lambda$, define $D(I) = \bigcap_{i \in I} A_i - \bigcup_{j \notin I} A_j$. We call $D(I)$ an atom of $\{A_\lambda;\ \lambda \in \Lambda\}$ to index I (if $I = \emptyset$, $\bigcap_{i \in I} A_i = X$ and $\bigcup_{i \in I} A_i = \emptyset$).

Theorem 5.1 For all sub - set classes $\{A_\lambda;\ \lambda \in \tilde{\Lambda} \subset \Lambda\}$, the result of arbitary unions, intersections, differences and complements of $A_\lambda (\lambda \in \tilde{\Lambda})$ can be the form of some $D(I)$s.

Theorem 5.2 Autocontinuity and null - additivity of c - fuzzy measure are equivalent.

Proof Theorem 4.2 is the necessity of this theorem. In the following, we prove sufficiency.

If u is null - additive and not autocontinuous, from Theorem 4.7, we know u does not have property(S). Therefore, there exists a well ordered set class $\{B_\lambda\}_{\lambda < c} \subset \mathcal{F}$ such that for all $\{B_{\lambda_s}\} \subset \{B_\lambda\}$, there hold

$$u(\overline{\lim_{s \to c}}\, B_{\lambda_s}) > 0$$

Now, we shall choice $\{B_{\lambda_s}\} \subset \{B_\lambda\}$ which satisfies $\lim_{s \to c} B_{\lambda_s}$ exists. If this kind of $\{B_{\lambda_s}\}$ exists, from $\lim_{\lambda \to c} u(B_\lambda) = 0$, we have $\lim_{s \to c} u(B_{\lambda_s}) = 0$.

From Theorem 4.1, it can be gotten that

$$0 < u(\overline{\lim_{s \to c}}\, B_{\lambda_s}) = u(\lim_{s \to c} B_{\lambda_s}) = \lim_{s \to c} u(B_{\lambda_s}) = 0$$

This is a contradiction.

From $\lim_{\lambda \to c} u(B_\lambda) = 0$, we know there exists $\{B_{\lambda^n}\}_{n<\omega}$ such that

$$\lim_{n \to \infty} u(B_{\lambda^n}) = 0$$

Note $\mathcal{B} = \{D(I) = \bigcap_{i\in I} B - \bigcup_{i\notin I} B \ ; D(I) \neq \emptyset$, I is infinite$\}$. It is easy to obtain that the power of $\mathcal{B}$ is not more than c and $\overline{\lim_{n\to\infty}} B_{\lambda_n} = \bigcup \mathcal{B}$.

Well order $\mathcal{B}$ into $\{D1, D2, \ldots, D\lambda, \ldots\}_{\lambda<c}$. Let $Di_1 = D1$, we can find a sequence $\{B^{(1)}_{\lambda_n}\} \subset \{B_{\lambda_n}\}$ such that all $B^{(1)}_{\lambda_n} \supset Di_1$.

Obviously,

$$Di_1 \subset \underline{\lim_{n\to\infty}} B^{(1)}_{\lambda_n} \subset \overline{\lim_{n\to\infty}} B^{(1)}_{\lambda_n} \subset Di_1 \cup \bigcup_{i_1<\lambda<c} D_\lambda$$

Let $T1 = \{\lambda > i_1 ;$ there exist infinite $B^{(1)}_{\lambda_n} \supset D_\lambda\}$. If $T1 = \emptyset$, then

$$Di_1 \subset \underline{\lim_{n\to\infty}} B^{(1)}_{\lambda_n} \subset \overline{\lim_{n\to\infty}} B^{(1)}_{\lambda_n} \subset Di_1$$

and $\lim_{n\to\infty} B^{(1)}_{\lambda_n}$ exists and the conclusion follows.

If $T1 \neq \emptyset$, let i_2 be the least number of T1 and similarily we can get $\{B^{(2)}_{\lambda_n}\} \subset \{B^{(1)}_{\lambda_n}\}$ such that all $B^{(2)}_{\lambda_n} \supset Di_2$. Therefore,

$$Di_1 \cup Di_2 \subset \underline{\lim_{n\to\infty}} B^{(2)}_{\lambda_n} \subset \overline{\lim_{n\to\infty}} B^{(2)}_{\lambda_n} \subset Di_1 \cup Di_2 \cup \bigcup_{i_1,i_2<\lambda<c} D_\lambda$$

Let $T2 = \{\lambda > i_1, i_2;$ there exist infinite $B^{(2)}_{\lambda_n} \supset D_\lambda\}$.

Repeating above process one by one, we have for one $n<\omega$, $Tn = \emptyset$ and $\lim_{m\to\infty} B^{(n)}_{\lambda_m}$ exists, the conclusion follows, or for all $n<\omega$, $Tn \neq \emptyset$.

If the later case happens, define

$$\{B^{(\omega)}_{\lambda_n}\} = \{B^{(1)}_{\lambda_1}, B^{(2)}_{\lambda_1}, \ldots\}$$

we can verify that

$$\bigcup_{n<\omega} Di_n \subset \underline{\lim_{n\to\infty}} B^{(\omega)}_{\lambda_n} \subset \overline{\lim_{n\to\infty}} B^{(\omega)}_{\lambda_n} \subset \bigcup_{n<\omega} Di_n \cup \bigcup_{i_1,i_2\cdots<\lambda<c} D_\lambda$$

Regarding $B^{(\omega)}_{\lambda_n}$ as $\{B^{(1)}_{\lambda_n}\}$, we can still repeat above process. Using induction principle (of general well ordered set), we can obtain for one $\lambda< c$, $T\lambda = \emptyset$ and the conclusion follows, or for all $\lambda < c$, $T\lambda \neq \emptyset$.

If the later case happens, then for all $\lambda < c$,

$$Di_1 \cup Di_2 \cup \ldots \cup Di_\lambda \subset \underline{\lim_{n\to\infty}} B^{(\lambda)}_{\lambda_n} \subset \overline{\lim_{n\to\infty}} B^{(\lambda)}_{\lambda_n} \subset Di_1 \cup Di_2 \cup \ldots \cup Di_\lambda \cup \bigcup_{i_1,i_2\cdots i_\lambda<\lambda<c} D$$

Let $\{B_{\lambda_s}\} = \{B^{(1)}_{\lambda_1}, B^{(2)}_{\lambda_1}, \ldots, B^{(\lambda)}_{\lambda_1}, \ldots\}_{\lambda<c}$. In this moment, the power of $\{B_{\lambda_s}\}$ may be c but not more than c and

$$Di_1 \cup Di_2 \cup \ldots \cup Di_\lambda \cup \ldots \subset \underline{\lim_{s\to c}} B_{\lambda_s} \subset \overline{\lim_{s\to c}} B_{\lambda_s} \subset Di_1 \cup Di_2 \cup \ldots \cup Di_\lambda \ldots$$

therefore, $\lim_{s\to c} B_{\lambda_s}$ exists and the conclusion follows.

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