INTEGRAL REPRESENTATION OF FUZZY POSSIBILITY MEASURES

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Short communication

Let $(X,\mathcal{A})$ be a measurable space and let $F(\mathcal{A})$ be a generated fuzzy σ-algebra, $F(\mathcal{A}) = \{\mu, \mu: X \longrightarrow [0,1], \mu \text{ is } \mathcal{A}\text{-measurable}\}$. In what follows we deal with Zadeh's fuzzy connectives. A fuzzy probability measure [2] m is a mapping, $m: F(\mathcal{A}) \longrightarrow [0,1]$, such that

(1) $m(0_X) = 1$ and $m(1_X) = 1$

(2) $\forall \mu, \eta \in F(\mathcal{A}): m(\mu \vee \eta) + m(\mu \wedge \eta) = m(\mu) + m(\eta)$

(3) $\forall \{\mu_n\} \subset F(\mathcal{A}), \mu_n \nearrow \mu: m(\mu_n) \nearrow m(\mu)$.

The simplest example of such fuzzy probability measure is due to Zadeh [5] ,

(4) $m(\mu) = \int_X \mu \, dP$

where P is a classical probability measure on $(X,\mathcal{A})$.

In general we have the following integral representation due to Klement [2] .

THEOREM 1. (Klement) m is a fuzzy probability measure on $(X,F(\mathcal{A}))$ if and only if there exists a unique probability measure P on $(X,\mathcal{A})$ and a P-almost surely determined $\mathcal{A}$-Markov kernel K such that

(5) $m(\mu) = \int_X K(x,[0,\mu(x)[)\,dP(x)$.

Of course for crisp events A we get $m(1_A) = P(A)$. Note that $K: X \times \mathcal{B}_{[0,1[} \longrightarrow R$ is called an $\mathcal{A}$-Markov kernel if it satisfies the following conditions:

(6) $\forall B \in \mathcal{B}_{[0,1[} : K(.,B): X \longrightarrow R$ is $\mathcal{A}$-measurable

(7) $\forall x \in X : K(x,.): \mathcal{B}_{[0,1[} \longrightarrow R$ is a probability measure on $\mathcal{B}_{[0,1[}$.

Zadeh in [6] introduced the notion of a possibility measure Π where addition is replaced by the supremum (maximum). Similarly, a fuzzy possibility measure Pos corresponds to a fuzzy probability measure (in the same way as a possibility measure corresponds to a probability measure), $Pos: F(\mathcal{A}) \longrightarrow [0,1]$,

(8) $Pos(0_X) = 0$ and $Pos(1_X) = 1$

(9) $\forall \{\mu_n\} \subset F(\mathcal{A})$: $Pos(\vee \mu_n) = \sup (Pos(\mu_n))$.

The simplest example of such fuzzy possibility measure is again due to Zadeh [6] ,

(10) $Pos(\mu) = \oint_X \mu \circ \Pi(\cdot)$

where $\oint$ is a Sugeno's integral (see e.g. [1]) and Π is a possibility measure on $(X,\mathcal{A})$. Here we suppose X be a finite or denumerable space. Recall that Sugeno's integral of μ is

(11) $\oint_X \mu \circ \Pi(.) = \sup_{a \in [0,1]} \min (a, \Pi(M_a))$

where $M_a = \{x \in X, \mu(x) \geq a\}$.

For X finite (denumerable) we have obtained the next result .

THEOREM 2. Pos is a fuzzy possibility measure on $(X,F(\mathcal{A}))$ if and only if there exists a possibility measure Π and a Π-$\mathcal{A}$-Markov kernel K such that

(12) $Pos(\mu) = \oint_X K(x,[0,\mu(x)[) \circ \Pi(.)$.

$K: X \times \mathcal{B}_{[0,1]} \longrightarrow R$ is called a Π-$\mathcal{A}$-Markov kernel if it satisfies the following conditions:

(13) $\forall\, B \in \mathcal{B}_{[0,1]} : K(\cdot,B) : X \longrightarrow R$ is $\mathcal{A}$-measurable

(14) $\forall\, x \in X : K(x,\cdot) : \mathcal{B}_{[0,1]} \longrightarrow R$ is a probability measure on $\mathcal{B}_{[0,1]}$

(15) $\int_X K(x,[0,1[)\cdot \Pi(\cdot) = 1$.

For the proof and more details see [4] . Note that we have some kind of the uniqueness in the Theorem 2. More concretly, for any fuzzy possibility measure Pos we have a unique possibility measure Π^* and a unique Π^*-$\mathcal{A}$-Markov kernel K^* such that

(16) for any $\Pi \geq \Pi^*$, (12) holds for K^* and Π

(17) for any $K \geq K^*$, (12) holds for K and Π^* if and only if for all $x \in X$, $K^*(x,[0,a[) < \Pi^*(\{x\})$ implies $K^*(x,[0,a[) = K(x,[0,a[)$.

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