The Permanent of L-Fuzzy Matrices and Some Applications

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In this note , we define the concept of the permanent *Per*(A), where A is a fuzzy matrix over a complete and distributive lattice L, and some properties of the permanent of L-fuzzy matrices are given. We will prove the expansion theorem for the permanent, it is an analogue of the Laplace expansion theorem for the determinants. And we obtain some necessary and sufficient conditions for Per (A) = 0 which is helpful for computation of rank of L-fuzzy matrices and computation of solutions of L-fuzzy relation equation. In the end of this paper, we discuss the relations between the permanent of L-fuzzy matrices and its regularity.

Keywords: Fuzzy matrix, L-Permanent, G-Inverse, L-Block matrix.

1. Permanent of L-Fuzzy Matrices

Throughout this paper $L = \langle L, \leqslant, \wedge, \vee \rangle$ (written L) always denotes a complete and distributive lattice. The greatest and the least elements of L are denoted by 1 and 0, respectively.

In this section, we define the permanent of L-fuzzy matrix and prove some elementary properties of L-permanent.

Definition 1.1. Let $A=(a_{ij})\in L^{m\times n}, m\leqslant n$. The permanent of A, written Per(A), is defined by

$$Per(A)=\bigvee_{\delta\in S}\{\bigwedge_{i=1,\cdots,m} a_{i\delta(i)}\},$$

where S is the set of all one to one mappings from $\{1,2,\ldots,m\}$ to $\{1,2,\ldots,n\}$.

It is obvious that the permanent is a mapping Per: $L^{m\times n}\rightarrow L$.

Proposition 1.1. Let $A,B\in L^{m\times n}, m\leqslant n$. then

(1) $A\leqslant B\Rightarrow Per(A)\leqslant Per(B)$;

(2) $Per(A\vee B)\leqslant Per(A)\vee Per(B)$;

(3) $\forall\ r\in L$, $Per(r\cdot A)=r\wedge Per(A)$, where $r\cdot A=(r\wedge a_{ij})\in L^{m\times n}$.

Proposition 1.2. Let $A\in L^{n\times n}$, then $Per(A)=Per(A^T)$.

Proposition 1.3. Let $A=(a_{ij})\in L^{m\times n}$, $m\leqslant n$. $P\in L^{m\times m}, Q\in L^{n\times n}$ be permutation matrices (i.e. every row and column has only one nonzero element 1), then $Per(A)=Per(PAQ)$.

Proposition 1.4. Let $A\in L^{m\times n}, m\leqslant n$, be a regular matrix, B is G-inverse of A, then $Per(AB)=Per(AB)^2$.

Proposition 1.5. Let $A=(a_{ij})\in L^{n\times n}$ be a symmetric matrix. If A is realizable, then $Per(A)=\bigwedge_{i=1}^{n} a_{ii}$.

In Proposition 1.6, 1.7, 1.8 below, let L be totally ordered lattice.

Proposition 1.6. If the rows of a regular matrix $A\in L^{m\times n}(m\leqslant n)$ form a standard basis of the row space R(A) (see [1]), then there exists a permutation matrix P such that $Per(A)=Per(PA)^2$.

Proposition 1.7. Suppose the nonzero rows of an idempotent matrix $E=(e_{ik})\in L^{n\times n}$ *form a standard basis of* $R(E)$, *then* $Per(E)=\bigwedge_{i=1}^{n} e_{ii}$.

Proposition 1.8. If $A=(a_{ij})\in L^{m\times n}$. $m\leqslant n$, then $Per(A)>0$ iff, there exists $\alpha\in L, \alpha>0$, such that $Per(A(\alpha))=1$, where $A(\alpha)$ is α—section matrix (see [1]).

Proof. "$\Rightarrow$". Assume $Per(A)>0$, then there is $\delta_0\in s$ such that $\bigwedge_{i=1}^{m} a_{i\delta_0(i)}>0$. *For* $\alpha=\bigwedge_{i=1}^{m} a_{i\delta_0(i)}$, $\bigwedge_{i=1}^{m} a_{i\delta_0(i)}^{(\alpha)}=1$, *where* $a_{i\delta_0(i)}^{(\alpha)}\in A(\alpha)$. *Hence* $Per(A(\alpha))=1$.

"$\Leftarrow$". Assume $\alpha \in L, \alpha > 0$, and $Per(A(\alpha)) = 1$, then there exists $\delta_0 \in S$, such that $\bigwedge_{i=1}^{m} a_{i\delta_0(i)}^{(\alpha)} = 1$, *i. e.* $\forall\ i, a_{i\delta_0(i)}^{(\alpha)} = 1,\ i = 1,2,\ldots m$. Hence $a_{i\delta_0(i)} > \alpha$, i. e. $Per(A) > 0$, This completes the proof.

2. Expansion Theorem of L-Permanent

In this section, we prove the expansion theorem of L-Permanent, it is not only important at the theory of fuzzy matrices, but helpful for the computing of permanent.

we shall require the following notations.

$$Let\ \Omega_{rn} = \{(\omega_1, \omega_2, \cdots, \omega_r) : 1 \leqslant \omega_1 \leqslant \omega_2 \cdots \leqslant \omega_r \leqslant n, \omega_i\ is\ integer,\ i = 1,2,\cdots,r\}$$

and for $A \in L^{m \times n}, \alpha \in \Omega_{rm}, \beta \in \Omega_{sn}$, let $A[\alpha|\beta]$ denote the $r \times s$ subma-trix whose (i,j) entry is $a_{\alpha_i \beta_j}$, $A(\alpha|\beta)$ denote the $(m-r) \times (n-s)$ submatrix of A comple-mentary to $A[\alpha|\beta]$— that is the submatrix obtained from A by deleting rows α and columns β.

Theorem 2. 1. If A is an $m \times n$ matrix, $2 \leqslant m \leqslant n$, and $\alpha \in \Omega_{rm}$, then

$$Per(A) = \bigvee_{\omega \in \Omega} [Per(A[\alpha|\omega]) \wedge Per(A(\alpha|\omega))]. \qquad (2.1)$$

In particular, for any $i, 1 \leqslant i \leqslant m$,

$$Per(A) = \bigwedge_{t=1}^{n} [a_{it} \wedge Per(A(i|t))]. \qquad (2.2)$$

Proof. Formula (2. 2) is a consequence of (2. 1), we prove only (2. 1). forany $\omega \in \Omega_{rn}$, we have

$$Per(A[\alpha|\omega]) \wedge Per(A(\alpha|\omega)) = [\bigvee_{\delta_1 \in S_1} (\bigwedge_{i=1}^{r} a_{\alpha_i \delta_1(\alpha_i)})] \wedge [\bigvee_{\delta_2 \in S_2} (\bigvee_{j=r+1}^{m} a_{\alpha_j \delta_2(\alpha_j)})],$$

where S_1 is set of one to one mappings from $(\alpha_1, \alpha_2, \cdots, \alpha_r)$ to $(\omega_1, \omega_2, \cdots, \omega_r)$, S_2 is set of one to one mappings from $(\alpha_{r+1}, \alpha_{r+2}, \cdots, \alpha_m)$ to $(\omega_{r+1}, \omega_{r+2}, \cdots, \omega_n)$, $(\alpha_1, \alpha_2, \cdots, \alpha_m)$, $(\omega_1, \omega_2, \cdots, \omega_n)$ are permutation of $(1,2,\cdots,m)$, $(1,2,\cdots,n)$ respectively. So that $\delta_1(\alpha_i) \neq \delta_2(\alpha_j), i = 1,2,\cdots,r,\ j = r+1,\ r+2,\cdots,m$. Since S_1, S_2, are finite sets and L is distributive lattice, therefore

$$Per(A) \geqslant Per(A[\alpha|\omega]) \wedge Per(A(\alpha|\omega)),$$

by the arbitrary of,

$$Per(A) \geqslant \bigvee_{\omega \in \Omega_{rm}} [Per(A[A|\omega]) \wedge Per(A(\alpha|\omega))] \tag{2.3}$$

Conversely, for any $\delta \in S$, S is set of one to one mappings from $(1,2,\cdots,m)$to $(1,2,\cdots,n)$, and

$$\bigwedge_{i=1}^{m} a_{i\delta(i)} = (\bigwedge_{i=1}^{r} a_{\alpha_i}(\alpha_i)) \wedge \bigwedge_{j=r+1}^{m} a_{\beta_j\delta(\beta_j)}),$$

where $(\alpha_1,\alpha_2,\cdots,\alpha_r,\beta_{r+1},\cdots,\beta_m)$ is a permutation of $(1,2,\cdots,m)$. Hence there exist $\omega_0 \in \Omega_{rm}, \delta_1 \in S_1$ and $\delta_2 \in S_2$, such that

$$\bigwedge_{i=1}^{m} a_{i\delta(i)} = [(\bigwedge_{i=1}^{r} a_{\alpha_i\delta_1(\alpha_i)}) \wedge (\bigwedge_{j=r+1}^{m} a_{\beta_j\delta_2(\beta_j)}],$$

therefore

$$\bigvee_{\omega \in \Omega_{rn}} [Per(A[\alpha|\omega]) \wedge Per(A(\alpha|\omega))] \geqslant Per(A). \tag{2.4}$$

By the (2.3) and (2.4), we have

$$Per(A) = \bigvee_{\omega \in \Omega_{rn}} [Per(A[\alpha|\omega]) \wedge Per(A(\alpha|\omega))].$$

that is (2.1) holds.

This completes the proof.

Corollary 2.2. Let

$$A = \begin{pmatrix} B & C \\ O & D \end{pmatrix}$$

be a $m \times n$ matrix, $m \leqslant n$, $B \in L^{s_1 \times t_1}, D \in L^{s_2 \times t_2}, s_1+s_2=m, t_1+t_2=n, s_i \leqslant t_i, i=1,2$. Then

$$Per(A) = Per(B) \wedge Per(D).$$

Remark. We can generalize the result by induction on members of block matrices of main diagonal line.

3. Some Conditions for Per(A)=0

In this section, we assume L is a totally ordered lattice (example, $L=[0,1]$).

Theorem 3.1. Let $A=(a_{ij}) \in L^{n \times n}$. Then $Per(A)=0$ iff A contains an $s \times t$ zero submatrix, and $s+t=n+1$.

Proof. "$\Rightarrow$" We prove by induction on n. Let $Per(A)=0$.

If $n=1$, then $A=0$.

Assume the condition is necessary for all square matrices of order less than n. If every entry of A is zero, the proof of necessary is obviously finished. Suppose that $a_{hk}>0$, then from Theorm 2. 1,

$$0 = Per(A) = \bigvee_{j=1}^{n} (a_{hj} \wedge Per(A(h|j))).$$

Since sum in the fuzzy algebra is supremum, hence $Per(A(h|k))=0$, by the induction hypothesis, we can find an $s_1 \times t_1$ zero submatrix of $A(h|k)$ such that $s_1+t_1=(n-1)+1=n$. Therefore there exist permutation matrices $P, Q \in L^{n\times n}$. such that

$$B = PAQ = \begin{pmatrix} X & 0 \\ Z & Y \end{pmatrix}$$

where $X \in L^{s_1 \times (n-t_1)}$, $O \in L^{s_1 \times t_1}$, $Z \in L^{(n-s_1)\times(n-t_1)}$, $Y \in L^{(n-s_1)\times t_1}$.

By Proposition 1. 3 and Corollary 2. 2, we have

$$0 = P\{er(B) = Per(A) = Per(X) \wedge Per(Y).$$

Hence without less of generality, let $Per(X)=0$, where X is $s_1 \times s_1$ square submatrix and $s_1<n$. By the induotion hypothesis, X contains uxv zero submtix such that $u+v=s_1+1$. Suppose $X[\alpha_1, \cdots, \alpha_u | \beta_1, \cdots, \beta_v]=0$, that is $B[\alpha_1, \cdots, \alpha_u | \beta_1, \cdots, \beta_v]=0$. We consider the submatrix

$$C = B[\alpha_1, \cdots, \alpha_u | \beta_1, \cdots, \beta_v, s_1 + 1, \cdots, n] ,$$

it is zero submatrix of A with u rows and $v+(n-s_1)$ column, moveover

$$\begin{aligned} u + v + (n - s) &= (u + v) + n - s_1 \\ &= s_1 + 1 + n - s_1 \\ &= n + 1. \end{aligned}$$

this completes the proof of necessary.

The proof of sufficient is obviously finished.

The following corollaries, whose proof is omitted, can be easily checked.

Corollary 3. 2. Let $A \in L^{m\times n}, m \leqslant n$. Then $Per(A)=0$ iff A contains $s \times t$ zero submatrix such that $s+t=n+1$.

Corollary 3. 3. Let $A \in L^{m\times n}, m \leqslant n$. Then $Per(A)=0$ iff$\forall\ \alpha \in L, \alpha>0$ $Per(A(\alpha))=0$. where $A(\alpha)$ is α-section matrix A. (c. f.) Proposition 1. 8).

4. Applications

In the, we discuss the relations between L-permanent and regularity of L-fuzzy matrix, where L is a totally ordered lattice.

Lemma 4. 1. (1) Let $A \in L^{m\times n}$, $A \in L^{n\times s}$ and A hasn't zero row (i. e. each entry of this row is zero). Then $X=0$, if $XA=0$;

(2) Let $A \in L^{s\times m}$, $X \in L^{m\times n}$ and A hasn't zero column (i. e. each entry of this column is zero). Then $X=0$, if $AX=0$.

Theorem 4. 2. Let $A \in L^{m\times n}$ be regular and

$$A = \begin{pmatrix} A_1 & B_1 \\ 0 & C_1 \end{pmatrix}$$

where A hasn't zero row, C hasn't zero column. Then A , C are regular and g-inverse of A has the form

$$B = \begin{pmatrix} \overline{A}_1 & \overline{B}_1 \\ 0 & \overline{C}_1 \end{pmatrix}$$

where A_1, C_1 are g-inverse of $\overline{A}_1, \overline{C}_1$, respectively.

Remark. The converse of the theorem fails to be true.

Corollary 4. 3. Let $A \in L^{m\times n}$, and

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_r \end{pmatrix}$$

Then:

(1) A is regular iff A_i is regular, $i=1,2,\cdots,r$;

(2) g-inverse of A has the form

$$B = \begin{pmatrix} \overline{A}_1 & 0 & \cdots & 0 \\ 0 & \overline{A}_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \overline{A}_r \end{pmatrix}$$

satisfying $\overline{A}_i$ is g-inverse of A_i, $i=1,2,\cdots,r$; the converse is true.

Theorem4. 4 Let $A\in L^{n\times n}$ be full rank and regular matrix. Then $Per(A)>0$. Proof. Assume $Per(A)=0$, by the Theorem 3. 1 there exist permutation matrices P, $Q\in L^{n\times n}$ such that

$$PAQ = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

where $0\in L^{s\times t}$ satisfying $s+t=n+1$. By the Theorem 4. 2, A_3 is regular. Therefore $\rho_r(A_3)=\rho_c(A_3)$, but $\rho_c(A_3)=n-t=s-1$ (because A is full rank), $\rho_r(A_3)=s$, this is a contration. Hence $Per(A)>0$.

Remark. If A is only full rank, the theorem isn't true.

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