ON OPERATIONS FOR PROBABILISTIC SETS.Part 1.

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This paper deals with theoretical aspects on operations for probabilistic sets and their distribution function representation. The considerations are illustrated by means of several examples.

Keywords: Probabilistic set, Triangular norm Distribution function, Averaging operator.

1. Introduction.

As known there are several investigations where attempt to combine fuzziness and randomness factors into one concept are considered [17,12,10,11,9]. Here we focus our attention on probabilistic sets in the sense of the work [9]. Logical connectives applied to probabilistic sets and applicational aspects of this sets in decision making processes were investigated by [3 - 6]. We are interested to get some generalization operation for probabilistic sets and distribution function representation of them. In consideration of the operations for probabilistic sets we shall use triangular norm (a t - norm for short), triangular conorm (t - conorm) and averaging operator concepts [13,15,7,8].

2. On probabilistic sets, and t-norms, t-conorms.

Let $(\Omega, \mathcal{B}, P)$ a probability space, U is universe, via $A(u,\omega)$ denotind the defining function $A: U \times \Omega \longrightarrow [0,1]$ of probabilistic set A. For each fixed $u \in U$ defining function is denoted by $A(u)$ and considered as the random variable on the probability space.

In this situation for characterization of probabilistic sets distribution function is introducing, i.e.

$$F_{A(u)}(z) = F_A(z) = P(\{\omega : A(u,\omega) < z\})$$

for each $z \in [0,1]$.

It is known [14,7] that t- norms (resp: t- conorms) provide a good model for fuzzy -set -theoretic intersections (resp: union). Therefore, let us recall the representation for t- norm and t- conorm [1,2]. Let two-place real function

$$H : \mathcal{J} \times \mathcal{J} \to \mathcal{J}, \quad \mathcal{J} = [a, b], \quad 0 \leq a < b \leq \infty$$

satisfies the following properties: a)associativity; b)commutativity; c)non-decreasing in each argument;d) $H(0,z) = z$.

Function $g : [0,1] \to [0,b]$ (resp: function $f : [0,1] \to [0,b]$) is a single-place continuously strictly increasing (resp: decreasing) one.

Theorem 2.1. The two-place real function

$$\perp : [0,1] \times [0,1] \to [0,1],$$

$$\perp(x_1, x_2) = g^{-1}(\min(g(1), H(g(x_1), g(x_2))))$$

is a t- conorm.

Theorem 2.2. The two-place real function

$$T : [0,1] \times [0,1] \to [0,1],$$

$$T(x_1, x_2) = f^{-1}(\min(f(0), H(f(x_1), f(x_2))))$$

is a t - norm.

Further, if $\top(x_1, x_2) = n(\perp(n(x_1), n(x_2)))$

and conversely $\perp(x_1, x_2) = n(\top(n(x_1), n(x_2)))$,

then $\top$, $\perp$ are said to be dual in regard to strong negation n [15,7].

Similarly, as in fuzzy sets theory various operation can be presented for probabilistic sets [3-6]. We shall use t- norm and t- conorm concepts in further consideration about operation for probabilistic sets. Thus, substituting of the arguments of t- norm (resp: t- conorm) $x_i \in [0, 1]$ for $A_i(u, \omega)$, intersection (resp: union) of probabilistic sets $A_1, A_2, \ldots, A_K$ will be given by equations:

$$\top(A_1, A_2, \ldots, A_K)(u, \omega) =$$

$$= f^{-1}(\min(f(0), G(f(A_1(u,\omega)), \ldots, f(A_K(u,\omega))))),$$

$$\perp(A_1, A_2, \ldots, A_K)(u, \omega) =$$

$$= g^{-1}(\min(g(1), G(g(A_1(u,\omega)), \ldots, g(A_K(u,\omega)))))$$

for all $u \in U$, $\omega \in \Omega$,

where G is a k- place semigroup operation on [o,b] having 0 as unit and properties a), b), c).

Now, as some particular cases we obtain

$$\top_\lambda(A_1, A_2, \ldots, A_K)(u, \omega) =$$

$$= f^{-1}(\min(f(0), \frac{1}{\lambda}(\prod_{i=1}^{K}(1 + \lambda f(A_i(u,\omega))/f(0)) - 1))),$$

$$\perp_\lambda (A_1, A_2, \ldots, A_K)(u,\omega) =$$
$$= g^{-1}\left(\min\left(g(1), \frac{1}{\lambda}\left(\prod_{i=1}^{K}(1+\lambda g(A_i(u,\omega))/g(1))-1\right)\right)\right)$$
for all $u \in U$, $\omega \in \Omega$.

3. Distribution function representation of operation based on t- norms and t- conorms.

Let us fix a point $u \in U$ and denote $A_i(u)$ by X_i . As known from probability theory

$$F_{\perp_K}(z) = \int\limits_{D_1(z)}\cdots\int \Psi_{X_1 \cdots X_K}(x_1, x_2, \ldots, x_K)\, dx_1 \ldots dx_K ,$$

$$F_{T_K}(z) = \int\limits_{D_2(z)}\cdots\int \Psi_{X_1 \cdots X_K}(x_1, x_2, \ldots, x_K)\, dx_1 \ldots dx_K ,$$

where $D_1(z)$ (resp: $D_2(z)$) is the region of integration being determined as

$$D_1(z) = \{(x_1, x_2, \ldots, x_K) : \perp(x_1, x_2, \ldots, x_K) < z\},$$

$$D_2(z) = \{(x_1, x_2, \ldots, x_K) : T(x_1, x_2, \ldots, x_K) < z\},$$

$\Psi_{X_1 \cdots X_K}(x_1, \ldots, x_K)$ is a joint probability density function of the vector $(X_1, X_2, \ldots, X_K)$,

$$\perp_K = \perp(X_1, X_2, \ldots, X_K), \quad T_K = T(X_1, X_2, \ldots, X_K).$$

Theorem 3.1. If $\perp_K, T_K$ are dual in regard to strong negation $n(x) = 1 - x$, then

$$F_{T_K}(z) + F_{\perp'_K}(1-z) = 1,$$

where $\perp'_K = \perp(1-X_1, \ldots, 1-X_K)$.

Proof. With the assumption of the theorem we obtain

$$F_{\perp'_K}(1-z) = P\left(\perp(1-X_1, 1-X_2, \ldots, 1-X_K) < 1-z\right) =$$

$$= P(T(X_1, X_2, \ldots, X_K) \geq z) = 1 - F_{T_K}(z).$$

Corollary 3.1. If $\perp_K, T_K$ are dual in regard to strong negation $n(x) = 1 - x$ and random vector $(X_1, X_2, \ldots, X_K)$ is uniformly distributed in the region $(0 \leq x_1 \leq 1, \ldots, 0 \leq x_K \leq 1)$ i.e.

$$\psi_{X_1 \cdots X_K}(x_1, \ldots, x_K) = \begin{cases} 1, & (0 \leq x_1 \leq 1, \ldots, 0 \leq x_K \leq 1), \\ 0, & \text{otherwize}. \end{cases}$$

then

$$F_{T_K}(z) + F_{\perp_K}(1-z) = 1.$$

Proof. Taking into account that if a random vector $(X_1, X_2, \ldots, X_K)$ is uniformly distributed ,then $(1-X_1, \ldots, 1-X_K)$ is also uniformly distributed, we can conclude that the following relationship holds true

$$F_{\perp'_K}(z) = F_{\perp_K}(z).$$

Example 3.1. Let

$$T_2 = T(X_1, X_2) = min\,(1, X_1 + X_2),$$

$$\perp_2 = \perp(X_1, X_2) = max\,(0, X_1 + X_2 - 1)$$

and vector (X_1, X_2) is uniformly distributed. In (6) has been proved that $F_{\perp_2}(z) = \frac{1}{2} z^2$.

So directly by the corollary 3.1.

$$F_{T_2}(z) = 1 - \frac{1}{2}(1-z)^2.$$

Further the distribution function of the as result of various operation on probabilistic sets will be founded by using formulas (5):

$$F_{\perp(X_1,X_2)}(z) = \int_0^z dx_1 \int_0^{h_1(z,x_1)} \psi_{X_1X_2}(x_1,x_2)\,dx_2\,,$$

$$F_{T(X_1,X_2)}(z) = \int_0^z dx_1 \int_0^1 \psi_{X_1X_2}(x_1,x_2)\,dx_2 + \int_z^1 dx_1 \int_0^{h_2(z,x_1)} \psi_{X_1X_2}(x_1,x_2)\,dx_2\,,$$

but here $x_2 = h_1(z,x_1)$ (resp: $x_2 = h_2(z,x_1)$) is determined from the equality $z = g^{-1}(H(g(x_1), g(x_2)))$ (resp: $z = f^{-1}(H(f(x_1), f(x_2)))$).

Let us consider the case when probabilistic sets $A_1, A_2, \ldots, A_k$ are independent (4), i.e. are such that for all point $u \in U$ correspondingly random variables are independent.

In this case

$$F_{\perp(X_1,X_2)}(z) = \int_0^z \psi_{X_1}(x_1)\, F_{X_2}(h_1(z,x_1))\,dx_1\,,$$

$$F_{\perp(X_1,\perp(X_2,X_3))}(z) = \int_0^z \psi_{X_1}(x_1) \left(\int_0^{h_1(z,x_1)} \psi_{X_2}(x_2) \, * \right.$$

$$\left. * \; F_{X_3}(h_1(h_1(z,x_1),x_2))\, dx_2 \right) dx_1 ,$$

which are special cases following the result

$$F_{\perp_K}(z) = \int_0^{t_1} \psi_{X_1}(x_1) \left(\cdots \left(\int_0^{t_{K-1}} \psi_{X_{K-1}}(x_{K-1}) F_{X_K}(t_K)\, dx_{K-1} \right) \cdots \right) dx_1 ,$$

where $t_1 = z$, $t_i = h_1(t_{i-1}, x_{i-1})$.

Similarly,

$$F_{T(X_1,X_2)}(z) = F_{X_1}(z) + \int_z^1 \psi_{X_1}(x_1) F_{X_2}(h_2(z,x_1))\, dx_1 ,$$

$$F_{T(X_1,T(X_2,X_3))}(z) = F_{X_1}(z) + \int_z^1 \psi_{X_1}(x_1) F_{X_2}(h_2(z,x_1))\, dx_1 +$$

$$+ \int_z^1 \psi_{X_1}(x_1) \left(\int_{h_2(z,x_1)}^1 \psi_{X_2}(x_2) F_{X_3}(h_2(h_2(z,x_1),\right.$$

$$\left. x_2))\, dx_2 \right) dx_1 ,$$

which are special cases of the result

$$F_{T_K}(z) = F_{X_1}(z) + \sum_{i=1}^{K-1} \int_{z_1}^1 \psi_{X_1}(x_1) \left(\int_{z_2}^1 \psi_{X_2}(x_2) \left(\cdots \right.\right.$$

$$\left.\left. \left(\int_{z_i}^1 \psi_{X_i}(x_i) F_{X_{i+1}}(z_{i+1})\, dx_i \right) \cdots \right) dx_2 \right) dx_1 ,$$

where $z_1 = z$, $z_i = h_2(z_{i-1}, x_{i-1})$.

Note, if this $\perp_K$, T_K are dual in regard to $n(x) = 1 - x$,then in accordance with theorem 3.1.

$$F_{T_K}(z) + F_{\perp'_K}(1-z) = 1.$$

The same results can be obtained directly. Indeed,first for $K=2$

$$F_{T(1-X_1, 1-X_2)}(z) = F_{1-X_1}(z) + \int_z^1 \psi_{1-X_1}(x_1) F_{1-X_2}(h_2(z, x_1)) dx_1 =$$

$$= 1 - \int_z^1 \psi_{1-X_1}(x_1) (1 - F_{1-X_2}(h_2(z, x_1))) dx_1 .$$

So far as $h_2(z, x_1) = 1 - h_1(1-z, 1-x_1)$, stating $s = 1 - x_1$ and taking into accound that $1 - F_{1-X_2}(1-z) = F_{X_2}(z)$, we have

$$1 - \int_z^1 \psi_{1-X_1}(x_1) (1 - F_{1-X_2}(1 - h_1(1-z, 1-x_1))) dx_1 =$$

$$= 1 - \int_z^1 \psi_{X_1}(1-x_1) F_{X_2}(h_1(1-z, 1-x_1)) dx_1 =$$

$$= 1 - \int_0^{1-z} \psi_{X_1}(s) F_{X_2}(h_1(1-z, s)) ds = 1 - F_{\perp(X_1, X_2)}(1-z).$$

Further $T'_K = T(1 - X_1, \dots, 1 - X_K) =$

$$= T(1 - X_1, T(1 - X_2, \dots, 1 - X_K)) = T(1 - X_1, 1 - \perp(X_2, \dots, X_K)).$$

Now,using the previous case one can write

$$F_{T'_K}(z) = 1 - \int_0^{1-z} \psi_{X_1}(x_1) F_{\perp(X_2, \dots, X_K)}(h_1(1-z, x_1)) dx_1 =$$

$$= 1 - F_{\perp_K}(1-z).$$

R e f e r e n c e s

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