COMPLEX FUZZY MEASURE

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1. Introduction

In the works [1] , [2] , [3] were analysed two types of fuzzy measures - an entropic and energetic one. Some further works deal about fuzzy measure from axiomatic point of view [4] , [5] .

Let $(X, \mathcal{S}, \nu)$ be a measurable space, $L_1(\nu)$ the set of ν - integrable functions defined on the X. In our paper we shall define a fuzzy set on the universum X as an element of a set

$$F(X) = \langle 0,1\rangle^X \cap L_1(\nu).$$

In order to get a better survey let us recall, that an entropic type measure h can be defined e.g. as follows:

$$h: F(X) \to \langle 0,+\infty) \quad , \quad f \mapsto h(f) = \Psi\left[\int \Upsilon(f)\, d\nu\right]$$

where $\Psi : \langle 0,+\infty) \to \langle 0,+\infty)$, $\Psi(0) = 0$, Υ is the map which satisfied the following conditions:

i/ $\Upsilon(0) = \Upsilon(1) = 0$

ii/ Υ is increasing on $\langle 0,1/2\rangle$

iii/ Υ is decreasing on $\langle 1/2,1\rangle$

One way how to define the measure φ of energetic type is the following one:

$$\varphi : F(X) \to \langle 0,+\infty) \quad , \quad f \mapsto \varphi(f) = \int f\, d\nu \ .$$

The above mentioned types of a fuzzy measure are usually studied independently. Our work is aimed to study these measures from the common point of view.

2. Fundamental definitions

Let h, φ have the above meaning. Let $\mathcal{K}$ be the set of the complex numbers. Then the map

$$m: F(X) \rightarrow \mathcal{K}\ ,\ f \mapsto m(f) = \varphi(f) + i\ h(f)$$

we shall call as a complex fuzzy measure.

Remark: Further we shall use the addition of the complex numbers only. Hence the set $\mathcal{K}$ can be replaced by the two-dimensional vector space.

In order to get a more complex conception on the properties of the studied fuzzy sets, let us introduce the following two maps:

$$R(f) = \frac{\varphi(f)}{h(f)}\ ,\quad h(f) \neq 0$$

$$A(f) = \frac{\nu(X)}{h(f)}\ ,\quad h(f) \neq 0$$

For the sake of simplicity let us assume $\nu(X) = 1$. The quantities R(f) and A(f) let us call the relative and abselute fuzziness of the fuzzy set f respectively.

3. Some spetial properties and fundamental interpretations

The quantities m,R,A are essentially dependent on the measures φ and h. If we put

$$\Psi(x) = x\ ,\ \psi(x) = x^p(1-x)^p\ ,\ p = 1,2,3,\dots$$

then it is possible to obtain especialy intresting relations. Further we show some properties for putting p = 1. The cases $p \geqq 2$ are considerably more complicated and only partially analogical. It is easy to see, that for all $f \in F(X)$ is valid:

$$1 = R(f) < A(f) < +\infty$$
$$R(f) + R(1-f) = A(f)$$
$$A(f) = A(1-f)$$

For our purposes let us define for f,g   F(X):

f,g are orthogonal ($f \perp g$) iff $f(x)g(x) = 0$ [ν a.e.].

Our complex measure m is σ-additive with respect to this type of orthogonality, i.e.

$$m\left(\sum_{i=1}^{\infty} f_i\right) = \sum_{i=1}^{\infty} m(f_i) \quad \text{for } f_i \in F(X) \; ; \; f_i \perp f_j \; , \; i \neq j \; .$$

In the case of replacing the above mentioned orthogonality condition by the weaker one

$$\sum_{i=1}^{\infty} f_i \leq 1$$

we have

$$m\left(\sum_{i=1}^{\infty} f_i\right) = \sum_{i=1}^{\infty} m(f_i) - 2i \sum_{\substack{i,j=1 \\ i \neq j}}^{\infty} \varphi(f_i f_j) \; .$$

Let us study the range $\mathcal{R}[m]$ of the complex fuzzy measure m. It is possible to show, that

$$\mathcal{R}[m] = \left\{ (x+iy) \in \mathcal{K} \; : \; 0 \leq x \leq 1 \; , \; 0 \leq y \leq x(1-x) \right\} \; .$$

The set $\mathcal{R}[m]$ enables to generate some fuzzy-sets structures of the fuzzy subsets of the set X. Let us denote

$$\mathcal{D} = \left\{ (x+iy) \in \mathcal{K} \; : \; y = x(1-x) \; , \; x \in \langle 0,1 \rangle \right\} \; .$$

It is possible to show, that $m^{-1}[\mathcal{D}]$ is the set of the constant functions $f \in F(X)$. From the algebraical point of view this set is a σ-lattice. If we replace the set $\mathcal{D}$ by the set

$$\mathcal{D}^{(\varepsilon)} = \left\{ z \in \mathcal{K} : \varrho\,(z, \mathcal{D}) < \varepsilon \right\} \cap R[m] \; ,$$

(ϱ is the usual metric on the complex plane $\mathcal{K}$), then $m^{-1}[\mathcal{D}^{(\varepsilon)}]$ is not a lattice any more. The validity of the lattice operations is damaged all the more the higher number $\varepsilon > 0$ is taken. Evidently $m^{-1}[\mathcal{D}^{(\varepsilon)}]$ "converges" to the usual lattice structure when $\varepsilon \to 0$. It is possible to do the analogical consideration by replacing the set $\mathcal{D}$ by the real unit interval $\langle 0,1 \rangle$ or by the simple set $\{0,1\}$.

The geometrical interpretation of the relative fuzzines is very simple. The fuzzy sets of the constant relative fuzzines are imaged (by the map m) on the lines passing through the point (0,0) of the complex plane.

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