MARTINGALE CONVERGENCE THEOREM IN F-QUANTUM SPACES

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In the paper the martingale convergence theorem is proved for sequences of F-observables in F-quantum spaces (see [8]). Recall that for compatible observables it was done in [7], for observables in quantum logics in [4], [1] and [5]. The main tool is a representation of F-observables by random variables given by Piasecki [6] and Dvurečenskij [2] and a variant of the Radon-Nikodym theorem [3].

We recall ([2]) that a fuzzy measurable space is a couple (Ω, M), where Ω is a nonemty set, and $M \subset [0, 1]^{\Omega}$ is soft fuzzy σ-algebra, i.e. (i) if $1(\omega) = 1$ for any $\omega \in \Omega$, then $1 \in M$; (ii) $a \in M$ implies $a^{\perp} := 1 - a \in M$; (iii) $\bigcup_{n=1}^{\infty} a_n := \sup_n a_n \in M$ whenever $(a_n)_n \subset M$; (iv) if $1/2(\omega) = 1/2$ for any $\omega \in \Omega$, then $1/2 \in M$.
In M is defined an unary operation $\perp : M \to M$, $a \mapsto a^{\perp}$, satisfying (i) $(a^{\perp})^{\perp} = a$ for any $a \in M$; (ii) if $a \leq b$, $a, b \in M$ then $b^{\perp} \leq a^{\perp}$.
By an F-observable of (Ω, M) we mean a mapping $x : B(R) \to M$ ($B(R)$ is the Borel σ-algebra of the real line R) such that (i) $x(E^c) = 1 - x(E)$, $E \in B(R)$; (ii) $x(\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} x(E_i)$, $(E_i)_i \subset B(R)$, where E^c denotes the set theoretical complement of E in R.

A P-measure is any mapping $m : M \to [0,1]$ such that $m(a \cup a^{\perp}) = 1$ for any $a \in M$; $m(\bigcup_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} m(a_n)$ whenever $(a_n)_n \subset M$, $a_i \leq 1 - a_j$ for $i \neq j$.
By a fuzzy probability space we mean any triplet (Ω, M, m), where is nonvoid set, M is a soft fuzzy σ-algebra and m is a P-measure.
The sum of any two F-observables x and y is defined as unique

F-observable $x + y$ such that $B_{x+y}(t) = \bigcup_{r \in Q} (B_x(r) \cap B_y(t - r))$, $t \in R$, where $B_x(r) = x((-\infty, r))$, $r \in Q$. Let h be a Borel function, then $h \circ x$ is an F-observable of (Ω, M) defined via $h \circ x : E \mapsto x(h^{-1}(E))$, $E \in B(R)$. The product of two F-observables x and y, $x.y$, is defined as follows $x.y = ((x + y)^2 - x^2 - y^2)/2$.

The mean value of an F-observable x in a P-measure m we mean $E(x) := \int_R t \, dm_x$, where $m_x : E \mapsto m(x(E))$, $E \in B(R)$, is a probability measure on $B(R)$, if the integral exists and is finite.

The indefinite integral, $\int_a x \, dm$, $a \in M$ is defined via $\int_a x \, dm := E(x.x_a)$, where x_a is the indicator observable of a fuzzy set $a \in M$ defined as follows:

$$x_a(E) = \begin{cases} a \cap a^{\perp} & \text{if } 0,1 \notin E \\ a & \text{if } 0 \notin E,\ 1 \in E \\ a^{\perp} & \text{if } 0 \in E,\ 1 \notin E \\ a \cup a^{\perp} & \text{if } 0,1 \in E. \end{cases}$$

Definitions and result

Definition 1. Let x, y be F-observables of (Ω, M), m be a P-measure. We shall say, that x, y are equal (x is less or equal to y) almost everywhere with respect to a P-measure m, we write $x = y$ a.e.[m], $x \leq y$ a.e.[m], respectively, if $m((x - y)(\{0\})) = 1$ ($m(y - x)([0, \infty)) = 1$).

Definition 2. Let x be an integrable F-observable on M, i.e. $x(E) \in M$ for every $E \in B(R)$, $M_0 \subset M$ be a fuzzy soft sub σ-algebra. Then by a version of conditional expectation of the F-observable x for M_0 ($E(x/M_0)$) we understand any F-observable y on M_0 with the property $\int_a y \, dm = \int_a x \, dm$ for every $a \in M_0$.

Definition 3. Let (Ω, M, m) be a fuzzy probability space, $(M_n)_n$

be a sequence of fuzzy soft σ-subalgebras of M, $(x_n)_n$ be a sequence of F-observables. Then the sequence $((x_n, M_n))_n$ will be called a submartingal if it holds:

1. $M_n \subset M_{n+1} \subset M$ for all n = 1,2,...;
2. x_n be the F-observable on M;
3. $x_n \leq E(x_{n+1}/M_n)$ a.e.[m], n = 1,2,....

Theorem. Let $((x_n, M_n))_n$ be a submartingal, $\sup_n E(|x_n|) < \infty$, then there is an F-observable $x : B(R) \to M_0$, where M_0 is the smallest fuzzy soft σ-algebra containing the union $\bigcup_{n=1}^{\infty} M_n$, and x_n converges to x a.e.[m].

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