Vagueness and Fuzzy Approach

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When recording the world, human being identifies objects and properties. The latter are phenomena accompanying objects and they lead to groupings of objects which need not be sharply (unambiguously) separated. The Alternative Set Theory (AST) has conveniet means to work with such groupings – see [5, 6].

Let φ be a property of objects x. Then the above considered grouping forms a class

$$X = \{x;\ \varphi(x)\}.$$

The phenomenon of vagueness raises as a consequence of the infinity phenomenon. By infinity, we mean the natural infinity which is encountered when recording a large variety of states.Large, classically finite, numbers may behave as infinite ones. A *set* (a sharply separated grouping) is infinite if it contains a proper class X (not being a set). In symbols,

$$X \subseteq a.$$

Then X is called a semiset. A typical semiset is a class of all the finite natural numbers. We denote it by $\mathbb{FN}$. There are infinite natural numbers α, i.e. such that

$$\mathbb{FN} \subseteq \alpha$$

holds true.

It is our requirement to measure the finiteness of numbers. Using fuzzy logic,
we obtain such a measure as folows: consider the implication

$$\mathbb{FN}(x) \Rightarrow \mathbb{FN}(x + 1)$$

(if x is finite then $x + 1$ is also finite). Quite naturally, this implication can hardly be entirely true. Hence, taking a certain infinite γ, we obtain a measure of finiteness of natural number α,

$$\hat{\mathbb{FN}}(\alpha) = 0 \vee (1 - \frac{\alpha}{\gamma})$$

(cf. [2, 3]).

A crucial role in the problems of the phenomenon of vagueness is played by the notion of indiscernibility. AST introduces the indiscernibility relation $\equiv$ as follows:

$$x \equiv y \quad \text{iff} \quad \langle x, y \rangle \in \bigcap \{R_n;\ n \in \mathbb{FN}\}$$

where each R_n is a sharp, reflexive and symmetric relation serving us as a criterion of indiscernibility of x and y. Hence, $x \equiv y$ holds true if all the attempts to discern x, y using the (sharp) criteria R_n, $n \in \mathbb{FN}$, fail.

A typical example of the indiscernibility is the indiscernibility of rational numbers defined by

$$x \doteq y \quad \text{iff} \quad |x - y| < \frac{1}{n}$$

for every $n \in \mathbb{FN}$. Since $\mathbb{FN}$ is a vague class, we may assume to have exactly

$$\{R_\beta;\ \beta \leq \gamma\}$$

criteria of discernibility. Then the amount ot our "effort" to discern the objects x, y is given by the number α of the criteria R_β for which

$$\langle x, y \rangle \in R_\beta$$

holds true. We say that x and y are equal in the degree ν,

$$x \equiv_\nu y,$$

if $\nu = \frac{\alpha}{\gamma}$.

Theorem 1. (a) $x \equiv_1 x$.

(b) $x \equiv_\nu y$ implies $y \equiv_\nu x$.

(c) $x \equiv_{\nu_1} y$ and $y \equiv_{\nu_2} z$ implies $x \equiv_{\nu_3} z$

where $\nu_1 \otimes \nu_2 \precsim \nu_3$.

The $\otimes$ is the operation of bold product (see e.g. [1]) and $\precsim$ is the relation defined on rational numbers by $x \precsim y$ iff $x \leq y$ or $x \doteq y$. Hence, $\equiv_\nu$ is a fuzzy equality in the sense of [4].

A class X is *real* if there is an indiscernibility relation $\equiv$ such that $x \in X$ and $x \equiv y$ implies $y \in X$.

As a hypothesis, we accept that vagueness of natural phenomena is a consequence of the indiscernibility phenomenon. Hence, we restrict ourselves only to real classes.

A *kernel* $Y \subseteq X$ is a class which is dense in X, i.e. to every $x \in X$ there is $y \in Y$ such that $\langle x, y\rangle \in R_n$ for some $n \in \mathbb{FN}$. Then we put

$$X^{\mathbf{F}}(x) = \sup \{\nu;\ (\exists y \in Y)(x \equiv_{\nu} y\}$$

and call it the membership degree of x into X. It is a measure of the intensity of our "effort" to discern x from some element of the kernel Y.

Theorem 2. (a) $y \in Y$ implies $X^{\mathbf{F}}(y) = 1$.

(b) $X^{\mathbf{F}}(x) > 0$ holds for every $x \in X$.

(c) $\mathbb{FN}^{\mathbf{F}}(\alpha) \doteq \hat{\mathbb{FN}}(\alpha)$.

Let us consider two classes X_1, X_2 which are real according to the indiscernibility relations $\equiv^{(1)}$ and $\equiv^{(2)}$, respectively. The basic operations with them lead to considering of special kinds of interaction between the members of the respective sequences

$$\{R^{(1)}_{\beta};\ \beta \leq \gamma\}$$

and

$$\{R^{(2)}_{\beta};\ \beta \leq \gamma\}.$$

Using certain fine deliberations, we obtain the following theorem:

Theorem 3.

(a) $X^{\mathbf{F}}_1(x) \otimes X^{\mathbf{F}}_2(x) \leq (X_1 \cap X_2)^{\mathbf{F}}(x) \leq X^{\mathbf{F}}_1(x) \wedge X^{\mathbf{F}}_2(x)$.

(b) $X^{\mathbf{F}}_1(x) \vee X^{\mathbf{F}}_2(x) \leq (X_1 \cup X_2)^{\mathbf{F}}(x) \leq X^{\mathbf{F}}_1(x) \oplus X^{\mathbf{F}}_2(x)$.

(c) $(\neg X)^{\mathbf{F}}(x) = (V - X)^{\mathbf{F}}(x) \doteq 1 - X^{\mathbf{F}}(x)$.

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