ON THE SUM OF OBSERVABLES IN QUANTUM SPACES

Beloslav RIEČAN, Liptovský Mikuláš

In [4] a new model for quantum mechanics was suggested (so-called F-quantum space) based on the fuzzy sets theory and analogous to the theory of quantum logics ([6]). While in the theory of quantum logics the sum of observables need not exists ([1]), in the F-quantum space theory the sum exists always ([3, 5]). In this note we consider a more general structure ([2]) including F-quantum spaces as well as quantum logics. In the framework we study the problem of the existence of the sum of observables.

Definitions

By a QF-lattice (= quasiorthocomplemented lattice [2]) we understand a σ-complete lattice M with the greatest element 1 and the least element 0 with a mapping $a \mapsto a'$ satisfying the following conditions: (i) $(a')' = a$ for every $a \in M$; (ii) $a \leq b \implies b' \leq a'$.

An observable is a σ-homomorphism $x: B(R) \longrightarrow M$ defined on the σ-algebra of all Borel subsets of the real line, i.e. a mapping satisfying the conditions: (i) $x(E') = (x(E))'$ for every Borel set E; (ii) $x(\bigcup_n B_n) = \bigvee_n x(E_n)$ for every sequence of Borel sets.

As a classical example a probability space (X, S, P) can be considered with M = S. To every random variable $f: X \longrightarrow R$ an observable $x: B(R) \longrightarrow M$ can be assigned by the formula $x(E) = f^{-1}(E)$. A more general case is an F-quantum space, i.e. a family M of fuzzy subsets g of a set X ($g: X \to \langle 0, 1\rangle$) such that (i) $1_X \in M$; (ii) $f \in M \implies 1 - f \in M$; (iii) $f_n \in M$ ($n = 1, 2, \dots$) $\implies \sup_n f_n \in M$. On the other hand every quantum logic is a QF-lattice. E.g. if M is the family of all subspaces of a given Hilbert space, then as $a \wedge b$ one can consider the intersection of the subspaces a and b, $a \vee b$ the subspace generated by a and b and a' the orthogonal complement of the space a.

Now we shall define the sum of two observables $x, y: B(R) \to M$. We say that the sum exists, if there is and observable $z: B(R) \to M$ such that

$$z((-\infty, t)) = \bigvee_{r \in Q} (x(-\infty, r) \wedge y(-\infty, t-r))$$

for every $t \in R$. The observable z is then called the sum of x and y and it is denoted by $z = x + y$.

As a motivation two random variables $f, g: X \to R$ can be considered. Then $f + g < t$ iff $f < t - g$. In this case one can choose $r \in Q$ such that $f < r < t - g$, i.e. $f < r$ and $g < t - r$. Therefore

$$\{u ; f+g\,(u) < t\} = \bigcup_{r \in Q} \{u ; f(u) < r, g(u) < t-r\} =$$
$$= \bigcup_{r \in Q} (f^{-1}(-\infty, r) \cap g^{-1}(-\infty, t-r)).$$

Results

Theorem 1. Let M be a σ-distributive QF-lattice (i.e. $a \wedge \bigvee_n a_n = \bigvee_n (a \wedge a_n)$ for every $a, a_n \in M$). Then there is the sum of every observables $x, y: B(R) \to M$.

The condition that M is σ-distributive can be substitute by a weaker one. Two elements $a, b \in M$ are called to be compatible ($a \longleftrightarrow b$), if $a = (a \wedge b) \vee (a \wedge b')$ and $b = (a \wedge b) \vee (b \wedge a')$. A σ-complete lattice M is called C-σ-distributive, if

$a \wedge (\bigvee_n a_n) = \bigvee_n (a \wedge a_n)$ whenever $a \leftrightarrow a_n$ for every n. Recall that every quantum logic is C-σ-distributive.

Theorem 2. Let M be a C-σ-distributive QF-lattice. Then for every compatible observables $x,y:B(R) \longrightarrow M$ (i.e. such that $x(E) \longleftrightarrow y(F)$ for every Borel sets E, F) there exists the sum $x + y$.

References

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