PROBABILITY, POSSIBILITY and FUZZY SETS : various facets of uncertainty

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Historically, it is well known that the notion of probability emerged in the 17th century as a dual concept : chance, related to gaming problems and subjective uncertainty, related to the question of reliability of testimonies. In the works of pioneers of probability theory, such as J. Bernoulli, chance, very soon connected to frequency of occurence, was an additive notion but subjective probability was not so. However with the development of physical sciences, the non-additive side of probability was forgotten. So much so as 20th century researchers in decision theory have devoted much effort in the non-frequentist justification of additive probability as a model for subjective uncertainty in rational decision strategies.

The advent of the computer age has recently produced numerous works whose focus is the formal representation of imperfect knowledge, with no particular emphasis on decision. These works seem to be a revival of nonadditive probability. They also bridge the gap between uncertainty and very old notions of modality (such as possibility and necessity) used by Aristotle and Middle-Ages scholars, notions which have been reintroduced with full strength in this century by modal logics.

The new uncertainty models can be classified into three families that do not address the same issues. A survey is in (Dubois and Prade, 1988).

A first trend is to incorporate imprecision in additive probability. This is the family of upper and lower probability functions, that include Shafer belief functions as a special case, although some scholars advocating this model reject the point of view of upper and lower probability. Nevertheless, upper and lower probability systems provide numerical counterparts to Aristotelian modalities.

The second trend denies that numerical uncertainty judgment are additive. The uncertainty of a disjunction of mutually exclusive events can be any combination of the uncertainties of these events, provided that this combination behaves in accordance to the properties of the set of events. The obtained non-additive models are called decomposable measures.

The third trend has acknowledged conceptual imprecision and lexical vagueness as an important source of uncertainty in subjective knowledge. Fuzzy sets have been proposed by Zadeh as simple models of vague categories, thus suggesting a natural interpretation for multiple-valued logics that were introduced in the thirties. Pieces of vague information described in terms of fuzzy sets induce measures of uncertainty named possibility and necessity measures whose definitions were pioneered by Shackle in the fifties. This trend emphasizes a set-theoretic point of view that is not so present in the two others.

These families of uncertainty models interfere at the mathematical levels. All models are special cases of set-functions that are monotonic with respect to logical entailment. Possibility and necessity measures based on fuzzy sets are special cases of upper and lower probabilities as well as decomposable measures.

The duality between frequentist and subjectivist views of uncertainty is still around. It is natural to use frequencies for repeatable events ; but in other cases uncertainty is more a matter of ordering events in terms of individual belief. The notion of possibility is also a dual concept according to whether it refers to the ease of attainment (physical possibility), or to the consistency with available knowledge.

The frequentist point of view is naturally captured by upper and lower probability models once the traditional experimental setting of statistics is modified so as to accomodate imprecise or vague observations, for instance. The decision-theoretic setting also leads to upper and lower probabilities if we accept the notion of partial bet (including the case when due to his lack of information an individual refuses to bet). Decomposable measures, as distorted probabilities can be justified at the measurement-theoretic level by slightly relaxing Savage axioms of comparative probability (e.g. Dubois and Prade (1989)).

The main contribution of the third trend (fuzzy sets) in uncertainty modeling at the epistemological level is to point out another duality of uncertainty: a state of belief can be described in terms of generalized sets as naturally as in terms of degrees of (generalized) probability. To claim that "Jane is not married" is to designate the set {single, cohabitant, divorced, widow} as the set of *possible* values for Jane's status, that is, a uniform *possibility* distribution is thus defined. As uniform possibility measures are equivalent to sets, general possibility measures are equivalent to fuzzy sets. More generally upper and lower probabilities are generalizations of sets as much as generalized probabilities. For instance belief functions can be viewed as random sets. This means that there is an algebraic dimension present in many non-additive probability systems that is absent from classical probability theory because probability measures are only random points (Dubois and Prade, 1986).

A consequence of this state of facts is the existence of several kinds of measures of information in the non-additive probability framework. Along with generalized indices of entropy that evaluate the state of disorder, there exist, among others, generalized cardinality indices that evaluate imprecision (e.g. Klir (1987)).

Clearly this enlarged setting for uncertainty modelling, where traditional probability is only a part of the picture (albeit an essential one), and set theory finds room as a basis for the other companion modalities (possibility and necessity), creates many challenging problems about inductive inference and its links to logical deduction.

Among pending questions :

- How to extend probabilistic notions of conditioning and independence ? - Is conditioning irreducibly linked to probability ?
- Are there alternative methodologies to Bayesian inference for belief updating ?
- What role can be played by information-theoretic indices in formalizing basic principles such as the one of minimal change in belief kinematics ?
- Can generalized set-theoretic operations be useful in the problem of evidence combination ?

References

Dubois D., Prade H. (1986) A set-theoretic view of belief functions : logical connectives and approximations by fuzzy sets. Int. J. of General Systems, 12, 193-226.

Dubois D., Prade H. (1988) Modelling uncertainty and inductive inference. Acta Psychologica, 68, 53-78.

Dubois D., Prade H. (1989) Fuzzy sets, probability and measurement. Europ. J. of Operational Research, 40, 135-154.

Klir G. (ed.) (1987) Special Issue on Measures of Information. Fuzzy Sets and Systems, 24(2).