A PRELIMINARY STUDY OF THE THEORY OF GREY TOPOLOGICAL GROUP

(A PRELIMINARY STUDY OF THE THEORY OF COMPLEX FUZZY T.G.)

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ABSTRACT: In this paper the definiton and their theorems of grey topological group are introduced. And on this basis, the relationship between grey topological group and fuzzy and general topological group is discussed. There are also studies of subgroup of grey topological group.

KEYWORDS: Grey topological space, grey continuous mapping and grey topological group.

I. INTRODUCTION

D.H.Foster defined fuzzy topological group in 1979. We gave the concepts and properties of grey topological space in[2]. We shall study the grey topological group on this basis.

Defintion 1: Let X be a group(general group, its operation written" · "), A and B be grey subsets in X. Then the upper and lower subordinate functions of A · B(simply writen as AB) and A^{-1} are defined by:

$\bar{\mu}_{AB}(x)=\sup_{y\cdot z=x}\min\{\bar{\mu}_A(y),\bar{\mu}_B(z)\}$, $\underline{\mu}_{AB}(x)=\sup_{y\cdot z=x}\min\{\underline{\mu}_A(y),\underline{\mu}_B(z)\}$,

$\bar{\mu}_{A^{-1}}(x)=\bar{\mu}_A(x^{-1})$, $\underline{\mu}_{A^{-1}}(x)=\underline{\mu}_A(x^{-1})$, $\forall x\in X$.

Definition 2: Let $x_{\bar{\lambda}\underline{\lambda}}$ and $y_{\bar{\alpha}\underline{\alpha}}$ be grey points of X, then

$x_{\bar{\lambda}\underline{\lambda}}\cdot y_{\bar{\alpha}\underline{\alpha}}=(x\cdot y)_{\min(\bar{\lambda},\bar{\alpha}),\min(\underline{\lambda},\underline{\alpha})}$, $(x_{\bar{\lambda}\underline{\lambda}})^{-1}=(x^{-1})_{\bar{\lambda}\underline{\lambda}}$, usually written $x^{-1}_{\bar{\lambda}\underline{\lambda}}$.

Definition 3: Let $(X_t,\mathfrak{T}_t)(t\in T)$ be a family of grey topological spaces, $X=\prod_{t\in T}X_t$ be cartesian product and $\forall t\in T$, f_t: $X\to X_t$ be projections. Then the grey topology with $\mathcal{B}=\{f_t^{-1}[B_t]\mid B_t\in\mathfrak{T}_t, t\in T\}$ as subbase is called grey product topology and $(X,\mathfrak{T})$ is called grey product topologcal space, written $(X,\mathfrak{T})=(X_1,\mathfrak{T}_1)\times(X_2,\mathfrak{T}_2)\times\cdots\cdots$.

Theorem 1: Let $(X_1,\mathfrak{T}_1)$ and $(X_2,\mathfrak{T}_2)$ be grey topological spaces W be a open coincidence field of the grey point $(x_{\bar{\lambda}\underline{\lambda}}, y_{\bar{\alpha}\underline{\alpha}})$ in the grey product topological space $(X_1,\mathfrak{T}_1)\times(X_2,\mathfrak{T}_2)=(X,\mathfrak{T})$. Then there exists a open coincidence field U of $x_{\bar{\lambda}\underline{\lambda}}$ and a open coincidence field V of $y_{\bar{\alpha}\underline{\alpha}}$ such that $\bar{\mu}_{UV}(x)\leqslant\bar{\mu}_W(x)$, $\underline{\mu}_{UV}(x)\leqslant\underline{\mu}_W(x)$, $\forall x\in X$.

II. CONCEPTION OF GREY TOPOLOICAL GROUP

Definition 4: Let X be a group, $\mathfrak{T}$ be a grey topology ($(X,\mathfrak{T})$ be a grey topological space) which satisfies following conditions:

(1) **mapping** f: $(X,\mathfrak{T})\times(X,\mathfrak{T})\to(X,\mathfrak{T})$, $(x,y)\mapsto x\cdot y$ is grey continuous.

(2) **mapping** g: $(X,\mathfrak{T})\to(X,\mathfrak{T})$, $x\mapsto x^{-1}$ is grey continuous.

Then $(X,\mathfrak{T})$ is called a grey topological group.

Theorem 2: The condition (1) of definition 4 is equivalent to (1)$'$.

(1)$'$ For any grey point $(x_{\bar{\lambda}\underline{\lambda}},y_{\bar{\alpha}\underline{\alpha}})\in(X,\mathfrak{T})\times(X,\mathfrak{T})$ and any coincidence field W of $x_{\bar{\lambda}\underline{\lambda}}\cdot y_{\bar{\alpha}\underline{\alpha}}$, there exists a coincidence field U of $x_{\bar{\lambda}\underline{\lambda}}$ and a coincidence field V of $y_{\bar{\alpha}\underline{\alpha}}$ such that

$$\bar{\mu}_{UV}(x)\leqslant\bar{\mu}_W(x),\quad \underline{\mu}_{UV}(x)\leqslant\underline{\mu}_W(x),\quad \forall x\in X.$$

Theorem 3: The condition (2) of definition 4 is equivalent to (2)$'$.

(2)$'$ For any $x_{\bar{\lambda}\underline{\lambda}}\in X$ and any coincidence field U of $x^{-1}_{\bar{\lambda}\underline{\lambda}}$, there exists a coincidence field V of $x_{\bar{\lambda}\underline{\lambda}}$ such that

$$\bar{\mu}_{V^{-1}}(x)\leqslant\bar{\mu}_U(x),\quad \underline{\mu}_{V^{-1}}(x)\leqslant\underline{\mu}_U(x),\quad \forall x\in X.$$

Theorem 4: The conditions (1) and (2) of definition 4 are equivalent to (3).

(3) For any $x_{\bar{\lambda}\underline{\lambda}}, y_{\bar{\alpha}\underline{\alpha}}\in X$ and any coincidence field W of $x_{\bar{\lambda}\underline{\lambda}}\cdot y^{-1}_{\bar{\alpha}\underline{\alpha}}$, there exists a coincidence field U of $x_{\bar{\lambda}\underline{\lambda}}$ and a coincidence field V of $y_{\bar{\alpha}\underline{\alpha}}$ such that

$$\bar{\mu}_{UV^{-1}}(x)\leqslant\bar{\mu}_W(x),\quad \underline{\mu}_{UV^{-1}}(x)\leqslant\underline{\mu}_W(x),\quad \forall x\in X.$$

Theorem 5: Let $(X,\mathfrak{T})$ be a grey topological group, then following **mappings** are grey **homeomorphic**.

(1) f_a: $X\to X$, $x\mapsto xa$;

(2) f_a': $X \to X$, $x \mapsto ax$;

(3) g: $X \to X$, $x \mapsto x$,

where a is the fixed general element in X.

Proof: We prove (1) only.

First of all we prove fa is the bijection.

Since X is a group, hence for any $x, y \in X$, if $xa=ya$, then $x=x(aa^{-1})=(xa)a^{-1}=(ya)a^{-1}=y(aa^{-1})=y$. Thus fa is the injection.

Also for any $y \in X$, $y=xa \to x=ya^{-1}$.

Hence $f_a(ya^{-1})=(ya^{-1})a=y(a^{-1}a)=y$. Thus fa is the surjection.

Then we prove fa is grey continuous.

Let $y=xa$, W is a grey open coincidence field of y.

From theorem 1 we have there exists a grey open coincidence field U of x and a grey open coincidence field V of y such that $\bar{\mu}_{UV}(x) \leq \bar{\mu}_W(x)$, $\underline{\mu}_{UV}(x) \leq \underline{\mu}_W(x)$, $\forall x \in X$.

Since $a \triangle V$, hence $\bar{\mu}_{Ua}(x) \leq \bar{\mu}_W(x)$, $\underline{\mu}_{Ua}(x) \leq \underline{\mu}_W(x)$, $\forall \in X$, where $Ua=\{z \cdot a \in X, z \triangle U\}$. From [4] we have fa is grey continuous.

At last in the same way we can also prove fa is grey continuous. Hence fa is grey homeomorphic.

Theorem 6: Let $(X, \mathfrak{J})$ be a grey topological group, A be a grey open(closed)subset in X. Then Aa, aA and A^{-1} are grey open(closed) subsets too in X, where a is any general element of X.

Proof: Since $Aa=f_a[A]$, $aA=f_a'[A]$ and $A^{-1}=g[A]$. Hence from theorem 5 we have f_a, f_a' and g are all grey continuous mappings. From [4] we have $f_a[A]$, $f_a'[A]$ and $g[A]$ are all grey open(closed) subsets. Hence Aa, aA and A^{-1} are all grey open(closed) subsets.

Corllary: Let $(X, \mathfrak{J})$ be a grey topological group, A be a grey open subset in X and $\tilde{A}$ be a nonempty Cantor subset of X. Then $A\tilde{A}$ and $\tilde{A}A$ are gey open subsets in X.

III. RELATIONSHIP BETWEEN GREY TOPOLOGICAL GROU AND FUZZY AND GENERAL TOPOLOGICAL GROUP

1. Let $(X, \mathfrak{J})$ be a grey topological group. If the upper and lower suborinate functions of the grey subsets A in X are

equal, or $\bar{\mu}_A(x) = \underline{\mu}_A(x)$, $\forall x \in X$, then change the grey subsets A into the fuzzy subsets. Hence change the grey topological space $(X, \mathfrak{J})$ into the fuzzy topological space and change the grey continuous mappings f and g into the fuzzy continuous mappings. Hence change the grey topological group $(X, \mathfrak{J})$ into the fuzzy topological group.

2. Let $(X, \mathfrak{J})$ be a fuzzy topological group. If the subordinate functions of the fuzzy subsets A, $\mu_A(x) \in \{0, 1\}$, $\forall x \in X$, then change the fuzzy subsets A into the Cantor subsets. Hence change the fuzzy topological space $(X, \mathfrak{J})$ into the general topological space and change the fuzzy continuous mappings f and g into the general continuous mappings. Hence change the fuzzy topological group $(X, \mathfrak{J})$ into the general topological group.

Thus general topological group is a particular example of fuzzy topological group. Fuzzy topological group is a particular example of grey topological group.

So grey topological group $\supseteq$ fuzzy topological group $\supseteq$ general topological group.

IV. SUBGROUP OF GREY TOPOLOGICAL GROUP

Definition 5: Let $(X, \mathfrak{J})$ be a grey topological group, S is a subgroup of the group X. Then $(S, \mathfrak{J}_S)$ is called the subgroup of the grey topological group $(X, \mathfrak{J})$, where $\mathfrak{J}_S = \{L \cap S \mid L \in \mathfrak{J}\}$.

If subgroup S is the grey open(closed) subset of the grey topological space $(X, \mathfrak{J})$, then $(S, \mathfrak{J}_S)$ is called the grey open (closed) subgroup of the grey topological group $(X, \mathfrak{J})$.

Theorem 8: A subgroup of grey topological group is the grey topological group.

Theorem 9: A grey open subgroup of grey topological group is the grey closed subgroup.

Definition 6: Let $(X, \mathfrak{J})$ be a grey topological group, S be a normal subgroup of the group X. Then $(S, \mathfrak{J}_S)$ is called the normal subgroup of the grey topological group $(X, \mathfrak{J})$, where $\mathfrak{J}_S = \{L \cap S \mid L \in \mathfrak{J}\}$.

Theorem 10: Let $(X, \mathfrak{J})$ be a grey topological group, e be a unit of the group X and A be a connected componet of the grey topological space $(X, \mathfrak{J})$. If $e \in A$, then $(A, \mathfrak{J}_A)$ is the normal subgroup of the grey topological group $(X, \mathfrak{J})$, where $\mathfrak{J}_A = \{L \cap A \mid L \in \mathfrak{J}\}$.

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