FUZZY QUANTUM LOGICS AND THE PROBLEM OF CONNECTIVES

J A R O S Ł A W   P Y K A C Z

Uniwersytet Gdański, Instytut Matematyki, ul. Wita Stwosza 57,
80-952 Gdańsk, Poland.

ABSTRACT: The problem which fuzzy connectives are the most appropriate for describing quantum systems in the language of fuzzy set theory is studied. It is shown that Giles bold union $\mu_{A\cup B}(x) = \min(1,\mu_A(x)+\mu_B(x))$ and intersection $\mu_{A\cap B}(x) = \max(0,\mu_A(x)+\mu_B(x)-1)$ are better for this purpose than Zadeh max and min operations.

I. Introduction.

As soon as quantum mechanics achieved its mature form it became obvious that the very structure of its experimentally verifiable propositions differs from the structure of propositions of classical mechanics [1,2]. It is generally agreed that in classical mechanics a set of propositions is a Boolean algebra while in quantum mechanics it should possess the main features of a lattice of closed subspaces of an infinite-dimensional Hilbert space. Therefore, it should be a partially ordered orthocomplemented σ-orthocomplete orthomodular set, usually briefly called *quantum logic*, admitting full set of probability measures. For those who are not familiar with these notions we remind the definitions.

Def.1. By a *quantum logic* (or simply a *logic*) throughout this paper we mean partially ordered, orthocomplemented, σ-orthocomplete orthomodular set, i.e. a partially ordered set (abbr. "poset") L in which

(i) the least element O and the greatest element I exist,
(ii) the orthocomplementation map $':L\to L$, such that $a''=a$, $a\vee a'=I$, and $a\le b \Rightarrow b'\le a'$ is admitted,
(iii) the least upper bound $\vee_i a_i$ of any sequence of elements $a_1,a_2,a_3,\ldots$ such that $a_i\le a_j'$ for $i\ne j$, exists in L,
(elements a,b such that $a\le b'$ are called *orthogonal* and are denoted $a\perp b$)
(iv) the orthomodular identity $a\le b \Rightarrow b = a\vee(a'\wedge b)$ holds.

We would like to warn the reader accustomed to the fuzzy set notation that throughout this paper $a\vee b$ and $a\wedge b$ denote, respectively, the least upper bound (join) and the greatest lower bound (meet) of elements $a,b\in L$ with respect to the given partial order $\le$ and that they do not denote Zadeh max and min fuzzy connectives.

Def.2. By a *probability measure* on a logic L we mean a map $m:L\to[0,1]$ such that $m(I) = 1$ and $m(\vee_i a_i) = \sum_i m(a_i)$ for any sequence of pairwise orthogonal elements. A set S of probability measures on L is called *full* iff $m(a)\le m(b)$ for all $m\in S$ implies $a\le b$.

Elements of a logic are usually called *propositions* or *properties* or *yes-no observables* and it is assumed that they represent properties of a physical system. Probability measures on a logic represent states of a physical system and therefore they are usually called *states* on a logic. If a is a proposition and m is a state then the number $m(a) \in [0,1]$ is interpreted as a probability of obtaining positive result in an experiment testing a property of a physical system represented by a when this system is in a state represented by m.

Since for any proposition a and for any state m the number $m(a)$ belongs to the unit interval, states can be treated as fuzzy subsets of an universum L, and conversely, propositions can be treated as fuzzy subsets of an universum S. The second possibility was mentioned in [2], developed previously in [3,4,5] and it is also the subject of the present paper.

II. Fuzzy set approach to quantum logics.

The approach developed in [3,4,5] is based on the following theorem of Mączyński [6,7].

Theorem.1.(Mączyński [7], proof in [6]).

(i) If L is a logic with a full set of probability measures S, then each $a \in L$ induces a function $\underline{a}: L \rightarrow [0,1]$ where $\underline{a}(m) = m(a)$ for all $m \in S$. The set of all such functions $\underline{L} = \{\underline{a} : a \in L\}$ satisfies the following condition :

Orthogonality Postulate: If $\underline{a}_1, \underline{a}_2, \ldots$ is a sequence of functions such that

$$\underline{a}_i + \underline{a}_j \leq 1 \text{ for } i \neq j, \tag{1}$$

then there exists $\underline{b} \in \underline{L}$ such that $\underline{b} + \underline{a}_1 + \underline{a}_2 + \ldots = 1$.

(It is assumed that for one-element sequences the condition (1) is always satisfied).

$\underline{L}$ equipped with the natural partial order: $\underline{a} \leq \underline{b}$ iff $\underline{a}(m) \leq \underline{b}(m)$ for all $m \in S$ and complementation $\underline{a}' = 1 - \underline{a}$ is isomorphic to L.

(ii) Conversely, if $\underline{L} \subseteq [0,1]^X$ is a set of functions for which the Orthogonality Postulate is satisfied then it is a logic with respect to the natural partial order and complementation. Every point $x \in X$ induces a probability measure m_x on $\underline{L}$ where $m_x(\underline{a}) = \underline{a}(x)$ for all $\underline{a} \in \underline{L}$ and the set $\{m_x : x \in X\}$ is full.

Due to the part (ii) of Theorem 1 we can adopt the following definition:

Def 2. By a *fuzzy quantum logic* we mean any family $\mathcal{L}$ of fuzzy sets which membership functions satisfy the Orthogonality Postulate.

Let us note that the adjective *fuzzy* in this definition indicates only that this logic consists of fuzzy sets. Partial order and orthocomplementation are in this logic nothing else than the standard fuzzy set inclusion and complementation. Orthomodularity and σ-orthocompleteness conditions, i.e. conditions (iv) and (iii) of Definition 1, are satisfied due to the part (ii) of Theorem 1. Fuzzy quantum logic is not a fuzzyfication of an ordinary quantum logic in the usual sense of the fuzzyfication procedure. Actually, due to the part (i) of Theorem 1, any quantum logic with a full set of probability measures can be represented in a form of a fuzzy quantum logic.

Since in a fuzzy quantum logic $\mathcal{L}$ the Orthogonality Postulate must be satisfied, $\mathcal{L}$ is certainly <u>not</u> a family of all fuzzy subsets of S and, therefore, join $\vee$ and meet $\wedge$ in $\mathcal{L}$ do not coincide with Zadeh max and min fuzzy connectives. The same conclusion follows from the definition of an orthocomplementation. Since for a genuine fuzzy subset A of an universum X there exists at least one point $x \in X$ such that $\max(\mu_A(x), \mu_{A'}(x)) \neq 1$, we see that the condition $A \vee A' = I$ would be fulfilled for $\vee$ being the pointwise maximum only if a quantum logic $\mathcal{L}$ consisted exclusively of crisp subsets of S. However, such a situation is impossible if elements of $\mathcal{L}$ are supposed to represent properties of a physical system: If different states m_1, m_2 of a physical system have to be distinguished experimentally there should be at least one property of this system, represented by $a \in L$ such that $m_1(a) \neq m_2(a)$. Since the set S of probability measures on a logic is assumed to be a convex set (see, e.g. [2]), even if $m_1(a)=0$ and $m_2(a)=1$ there exists $m=pm_1+(1-p)m_2$ with $0<p<1$ so $m(a)=1-p\neq 0,1$ and the property a cannot be described by a crisp subset of S.

III. The problem of connectives.

Since we have seen that it is not possible to use Zadeh min and max connectives as meet and join in a fuzzy quantum logic $\mathcal{L}$, the question arises if there are other fuzzy connectives more suitable for this purpose. We meet here, in the very beginning, the following difficulty: When we look at the list of connectives most frequently used in the fuzzy literature (see, for example, [8,9]) we notice that all of them are defined in a pointwise manner. On the contrary, meet and join in a partially ordered set are "global" notions, i.e. even if we deal with a poset of functions, usually we cannot say what is the value of $(f \vee g)(x)$ and $(f \wedge g)(x)$ when we know only $f(x)$ and $g(x)$. Generally, to find $(f \vee g)(x)$ and $(f \wedge g)(x)$ we should know a whole partially ordered set of functions. However, when a quantum logic is translated with the aid of Mączyński Theorem into the language of fuzzy set theory, it is not merely a poset but it has a definite structure imposed by the Orthogonality Postulate. Already in Mączyński proof of his Theorem we can find that if $\underline{a}_1, \underline{a}_2, \ldots$ is a sequence of functions such as described in the Orthogonality Postulate then their meet $\vee_i \underline{a}_i$ exists, and

$$\vee_i \underline{a}_i = \sum_i \underline{a}_i . \tag{2}$$

Now, let us notice that if $\underline{a}_i, \underline{a}_j$ are membership functions of the fuzzy subsets A_i, A_j of the universum S, i.e. if $\underline{a}_i = \mu_{A_i}$, $\underline{a}_j = \mu_{A_j}$, the condition

$$\underline{a}_i + \underline{a}_j \leq 1 \tag{3}$$

can be expressed with the aid of Giles [10] *bold intersection*

$$\mu_{A_i \cap A_j}(x) = \max(0, \mu_{A_i}(x) + \mu_{A_j}(x) - 1) \tag{4}$$

in the following way

$$A_i \cap A_j = \emptyset. \tag{5}$$

Two fuzzy sets satisfying (5) are called *weakly disjoint* by Giles in [10]. If A_i and A_j are weakly disjoint elements of a fuzzy quantum logic

$\mathcal{L}$, isomorphic, respectively, to $a_i, a_j \in L$, then, according to (2) and to Mączyński Theorem, $a_i \vee a_j$ is isomorphic to a fuzzy subset of S which membership function is equal to the algebraic sum $\mu_{A_i} + \mu_{A_j}$. Since for weakly disjoint fuzzy sets A and B an algebraic sum of their membership functions coincides with the membership function of their *bold union* A B defined by Giles in [10] by

$$\mu_{A \boldsymbol{\cup} B}(x) = \min(\mu_A(x)+\mu_B(x),1), \tag{6}$$

we can see that in a fuzzy quantum logic $\mathcal{L}$

$$\text{if} \quad A_i \boldsymbol{\cap} A_j = \emptyset \quad \text{then} \quad A_i \vee A_j = A_i \boldsymbol{\cup} A_j \ . \tag{7}$$

By Mączyński Theorem the same holds for any sequence of pairwise weakly disjoint elements :

$$\text{if } A_i \boldsymbol{\cap} A_j = \emptyset \text{ for } i \neq j, \text{ then } \vee_i A_i = \boldsymbol{\bigcup}_i A_i \tag{8}$$

and

$$\mu_{\vee_i A_i} = \textstyle\sum_i \mu_{A_i} \ . \tag{9}$$

Therefore, the join of a sequence of pairwise weakly disjoint elements of a fuzzy quantum logic $\mathcal{L}$ can be obtained in a pointwise manner.

Giles bold union and intersection have also other nice features which cause that when we use them instead of Zadeh max and min connectives the resulting formulas are sometimes formally identical with respective formulas of the algebra of crisp sets. Since in the case of crisp sets Giles connectives (as well as Zadeh connectives) coincide with the set-theoretic operations, any formula in which they are used holds for crisp sets as well, so we obtain a nice generalization of ordinary set-theoretic calculus. Therefore, in the sequel, when the word "set" is used it can be either crisp or fuzzy.

First two useful identities, which poset-theoretic counterparts should necessarily hold in any quantum logic and which are not fulfilled by Zadeh operations, were mentioned already by Giles in [10].

Lemma.1. For any subset A of X

$$A \boldsymbol{\cap} A' = \emptyset \tag{10}$$

$$A \boldsymbol{\cup} A' = X. \tag{11}$$

Let us now consider the following crisp formula

$$A \subset B \quad \text{iff} \quad A \cap B' = \emptyset \ . \tag{12}$$

If we replace set-theoretic complement by the standard fuzzy complement and set-theoretic intersection by Zadeh min connective we obtain only the implication

$$\min(\mu_A, 1-\mu_B) = 0 \ \Rightarrow \ A \subset B \ , \tag{13}$$

but the opposite implication does not hold for non-crisp sets. On the contrary, if we replace set-theoretic intersection in (12) by Giles bold intersection, we obtain

$$\begin{aligned} A \boldsymbol{\cap} B' = \emptyset \ \text{ iff for all } x \quad & \max(0,\mu_A(x)+1-\mu_B(x)-1) = \\ & = \max(0,\mu_A(x)-\mu_B(x)) = 0, \end{aligned} \tag{14}$$

which holds iff $A \subset B$. Thus, we have proved the following lemma :

Lemma.2. For any two sets A,B

$$A \subset B \text{ iff } A \cap B' = \emptyset \ . \tag{15}$$

With the aid of this lemma we obtain immediately the following

Lemma.3. For any two sets A,B

$$A = B \text{ iff } A \cap B' = \emptyset \text{ and } B \cap A' = \emptyset \ . \tag{16}$$

Let us mention that the analogous useful formula holds in any Boolean algebra, provided that we replace bold intersection and fuzzy complement by their Boolean counterparts.

Finally, let us note that Zadeh connectives are distributive while Giles connectives are not. In the domain of quantum logics lack of distributivity is by no means a drawback, on the contrary, it is a virtue since lattices of closed subspaces of Hilbert spaces, contrary to Boolean algebras, are not distributive.

Bold union and intersection studied by Giles in [10] are fuzzy set counterparts of some of many-valued logic operations of Łukasiewicz and Tarski [11]. The first comparison of these operations and quantum logic operations was made by Frink already in 1938 [12].

BIBLIOGRAPHY

[1] Birkhoff, G. and von Neumann, J., Annals of Mathematics, 37 (1936), 823-843.

[2] Beltrametti, E.G. and Cassinelli, G., The Logic of Quantum Mechanics, Addison-Wesley, Reading 1981.

[3] Pykacz, J., Quantum logics as families of fuzzy subsets of the set of physical states, Preprints of the Second IFSA Congress, Tokyo, July 20-25 1987, vol. 2, 437-440.

[4] Pykacz, J., Quantum logics and soft fuzzy probability spaces, BUSEFAL, 32 (1987), 150-157.

[5] Pykacz, J., Probability measures in the fuzzy set approach to quantum logics, in : Proceedings of the First Winter School on Measure Theory, Liptovský Ján, January 10-15, 1988. A. Dvurečenskij and S. Pulmannová, eds. Bratislava, 1988.

[6] Mączyński, M. J., Int. J. Th. Phys. 8 (1973), 353-360.

[7] Mączyński, M. J., Int. J. Th. Phys. 11 (1984), 149-156.

[8] Trillas, E., Alsina, C. and Valverde, L., Do we need max, min and 1-j in fuzzy set theory, in : Fuzzy Set and Possibility Theory. Recent Developments, R. Yager, ed. Pergamon Press, New York, 1982.

[9] Weber, S., Fuzzy Sets and Systems, 11 (1983), 115-134.

[10] Giles, R., Int. J. Man-Machine Studies, 8 (1976), 313-327.

[11] Łukasiewicz, J. and Tarski A., Comptes Rendus Warsaw, Classe III, 23, (1930), 1-21.

[12] Frink, O., Amer. Math. Monthly 45 (1938), 210-219.

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