A NOTE ON A REPRESENTATION OF FUZZY OBSERVABLES

A. Kolesárová
Katedra matematiky EF SVŠT
Ilkovičova 3
812 19 Bratislava

R. Mesiar
Katedra matematiky SvF SVŠT
Radlinského 11
813 68 Bratislava
Czechoslovakia

Abstract

A representation theorem of fuzzy observables in fuzzy measurable spaces is presented in [1] . The main tool of [1] is the Loomis-Sikorski theorem. Another acces, much more simpler, utilizing the structure of fuzzy observables, has been presented in [2]. In this paper, we complete the results of [2].

1. Introduction

First, we recall the definitions of the basic notions.

Definition 1.

A fuzzy quantum space is a couple (Ω, M), where Ω is a non-empty set and $M \leq \langle 0,1 \rangle^{\Omega}$ is a system of fuzzy subsets of Ω such that :

(i) if $1_{\Omega}(\omega) = 1$ for any $\omega \in \Omega$, then $1_{\Omega} \in M$

(ii) if $u \in M$, then $u' = 1 - u \in M$

(iii) if $u_n \in M$, $n = 1,2,\ldots$, then $\bigvee_{n=1}^{\infty} u_n = \sup_n u_n \in M$

(iv) if $\left(\frac{1}{2}\right)_{\Omega}(\omega) = \frac{1}{2}$ for any $\omega \in \Omega$, then $\left(\frac{1}{2}\right)_{\Omega} \notin M$.

Definition 2.

Let (Ω, M) be a fuzzy quantum space. A mapping $x: \mathcal{B} \rightarrow M$ is said to be a fuzzy observable of (Ω, M), if :

(i) $x(E^c) = 1 - x(E)$ for any $E \in \mathcal{B}$

(ii) if $E_n \in \mathcal{B}$, $n = 1,2,\ldots$, then $x\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigvee_{n=1}^{\infty} x(E_n)$.

By the letter R we denote a set of all real numbers, by $\mathcal{B}$ a set of all Borel subsets in R and E^c denotes a complement of a set E in R.

According to [4] we denote by K(M) the set of all crisp subsets A of Ω for which there exists a fuzzy subset $u \in M$ such that :

$$\{\omega \in \Omega,\ u(\omega) > \tfrac{1}{2}\} \subseteq A \subseteq \{\omega \in \Omega,\ u(\omega) \geq \tfrac{1}{2}\} .$$

It is known (see [4]) that K(M) is a σ-algebra of crisp subsets of Ω .

2. The structure of fuzzy observables

According to [3] , let us introduce a function $x(\omega,.) : R \longrightarrow \langle 0,1 \rangle$ defined by $x(\omega,t) = x((-\infty,t))(\omega)$. Here ω is an arbitrary, but fixed element of Ω .

From the properties of a fuzzy observable x ((i) and (ii) of the Definition 2) it follows :

$$x(\omega,t) = \begin{cases} 1 - x(R)(\omega) & t \leq a_\omega \\ x(R)(\omega) & t > a_\omega \end{cases} \qquad (2.1.)$$

(see also [3]), where a_ω is a real number.

If $x(R)(\omega) > \frac{1}{2}$, then a_ω is determined by x and ω uniquelly. In the case $x(R)(\omega) = \frac{1}{2}$, a_ω can be chosen arbitrarily.

Now, let $E = (-\infty, a_\omega\rangle$. Then $E = \bigcap_{n=1}^{\infty}(-\infty, a_\omega + \frac{1}{n})$.

Since x is a σ-homomorphism , we get

$$x(E) = \bigwedge_{n=1}^{\infty} x((-\infty, a_\omega + \tfrac{1}{n})) .$$

As $x((-\infty, a_\omega + \frac{1}{n}))(\omega) = x(R)(\omega)$, we have $x(E)(\omega) = x(R)(\omega)$. On the other hand, $x(E)(\omega) = x((-\infty,a_\omega))(\omega) \vee x(\{a_\omega\})(\omega)$ and moreover, $x((-\infty,a_\omega))(\omega) = 1 - x(R)(\omega)$. These facts imply $x(\{a_\omega\})(\omega) = x(R)(\omega)$.

Now, let E be any set in $\mathcal{B}$. If $a_\omega \in E$, then since E can be expressed in the form $E = (E - \{a_\omega\}) \cup \{a_\omega\}$ and x is a fuzzy observable, we have $x(E)(\omega) = x(E - \{a_\omega\}) \vee x(\{a_\omega\})$. If we take into account that $x(E) \leq x(R)$ for any $E \in \mathcal{B}$ and the previous result $x(\{a_\omega\})(\omega) = x(R)(\omega)$, we get $x(E)(\omega) = x(R)(\omega)$.

In the other case, if $a_\omega \notin E$, we have $E \cap \{a_\omega\} = \emptyset$ and apply-

ing the properties of a fuzzy observable x we get
$x(E)(\omega)\wedge x(\{a_\omega\})(\omega) = x(\emptyset)(\omega)$ or $x(E)(\omega)\wedge x(R)(\omega) = 1-x(R)(\omega)$. Hence $x(E)(\omega) = 1 - x(R)(\omega)$. Thus the previous results can be written in the form :

$$x(E)(\omega) = \begin{cases} 1 - x(R)(\omega) & a_\omega \notin E \\ x(R)(\omega) & a_\omega \in E \end{cases} \qquad (2.2.)$$

3. Representation of fuzzy observables by random variables

Due to (2.1.) we can define a function

$$f : \Omega \longrightarrow R\ , \quad f(\omega) = a_\omega \qquad (3.1.)$$

Proposition 1.

The function $f : \Omega \longrightarrow R$ given by (3.1.) is a random variable on the space $(\Omega, K(M))$.

Proof. Let E be any set in $\mathcal{B}$ and let $\omega \in \Omega$ be such element that $x(E)(\omega) > \frac{1}{2}$. Then due to (2.2.) and the fact $1-x(R)(\omega) \leq \frac{1}{2}$ for any $\omega \in \Omega$, it holds $x(E)(\omega) = x(R)(\omega)$. But this implies $f(\omega) \in E$, i.e. $\omega \in f^{-1}(E)$. So we have shown that

$$\{\omega\ ,\ x(E)(\omega) > \tfrac{1}{2}\} \subseteq f^{-1}(E).$$

Conversely, let $\omega \in f^{-1}(E)$. Then $f(\omega) \in E$ and from (2.2.) we get $x(E)(\omega) = x(R)(\omega)$. Since $x(R)(\omega) \geq \frac{1}{2}$ for any $\omega \in \Omega$, it also holds $x(E)(\omega) \geq \frac{1}{2}$. Thus $f^{-1}(E) \subseteq \{\omega,\ x(E)(\omega) \geq \frac{1}{2}\}$ for any $E \in \mathcal{B}$. So we have proved that

$$\{\omega\ ,\ x(E)(\omega) > \tfrac{1}{2}\} \subseteq f^{-1}(E) \subseteq \{\omega\ ,\ x(E)(\omega) \geq \tfrac{1}{2}\} \qquad (3.2.)$$

holds for any $E \in \mathcal{B}$. (3.2.) together with the fact $x(E) \in M$ imply the $K(M)$-measurability of the function f. ///

The main result of Dvurečenskij in [1] is a representation theorem for fuzzy observables by random variables. We shall show that the random variable f given by (3.1.) is in fact Dvurečenskij´s representation of a fuzzy observable x.

Theorem 1.

Let x be a fuzzy observable of a fuzzy quantum space (Ω, M). Then

(i) There exists a random variable f on the space $(\Omega, K(M))$ such that (3.2.) , i.e.
$$\{\omega, x(E)(\omega) > \tfrac{1}{2}\} \subseteq f^{-1}(E) \subseteq \{\omega, x(E)(\omega) \geq \tfrac{1}{2}\}$$
holds for any $E \in \mathcal{B}$.

(ii) If $g : \Omega \longrightarrow R$ is any $K(M)$-measurable function satisfying (3.2.), then
$$\{\omega, f(\omega) \neq g(\omega)\} \subseteq \{\omega, x(R)(\omega) = \tfrac{1}{2}\} .$$

Proof. The statement (i) is an immediate consequence of Proposition 1.

Let us suppose that both f and g satisfy (3.2.). Let ω^* be any element of Ω for which $f(\omega^*) \neq g(\omega^*)$. Let us denote $E = \{f(\omega^*)\}$. Evidently $\omega^* \in f^{-1}(E)$. Therefore by (3.2.) we have $\omega^* \in \{\omega, x(E)(\omega) \geq \frac{1}{2}\}$, i. e. $x(E)(\omega^*) \geq \frac{1}{2}$.

On the other hand, since $f(\omega^*) \neq g(\omega^*)$, we have $g(\omega^*) \notin E$, i.e. $\omega^* \notin g^{-1}(E)$. Thus by (3.2.) $\omega^* \notin \{\omega, x(E)(\omega) > \frac{1}{2}\}$. It means $x(E)(\omega^*) \leq \frac{1}{2}$. Summarizing the previous results we obtain $x(E)(\omega^*) = \frac{1}{2}$. So the statement (ii) is proved. ///

Now, let us consider a random variable f on the space $(\Omega, K(M))$. Since f is $K(M)$-measurable, for any set E_r, $E_r = (-\infty, r)$, $r \in Q$ / Q is the set of all rational numbers /, there exists a fuzzy set $a_r \in M$ such that :
$$\{\omega, a_r(\omega) > \tfrac{1}{2}\} \subseteq f^{-1}(E_r) \subseteq \{\omega, a_r(\omega) \geq \tfrac{1}{2}\} \qquad (3.3.)$$

Let $\mu = \bigwedge_{r \in Q} (a_r \vee a_r')$. It is clear, that μ belongs to M and $\mu \geq \frac{1}{2}$. Further, for any $r \in Q$ let
$$\mu_r = (a_r \wedge \mu) \vee \mu' \qquad (3.\ 4.)$$
It is easy to see, that $\mu_r \in M$ and $\mu_r \vee \mu_r' = \mu$.

Moreover, it holds :

$$\{\omega , \mu_r(\omega) > \tfrac{1}{2}\} \subseteq f^{-1}(E_r) \subseteq \{\omega , \mu_r(\omega) \geq \tfrac{1}{2}\} \qquad (3.5.)$$

for any $r \in Q$.

Indeed, if we consider $\omega \in \Omega$ for which $\mu_r(\omega) > \frac{1}{2}$, then by (3.4.) and the fact $\mu'(\omega) \leq \frac{1}{2}$ we get $a_r(\omega) > \frac{1}{2}$. It means

$$\{\omega , \mu_r(\omega) > \tfrac{1}{2}\} \subseteq \{\omega , a_r(\omega) > \tfrac{1}{2}\} \qquad (3.6.)$$

If we consider $\omega \in \Omega$ for which $a_r(\omega) \geq \frac{1}{2}$, then also $a_r(\omega) \wedge \mu(\omega) \geq \frac{1}{2}$ and by (3.4.) $\mu_r(\omega) \geq \frac{1}{2}$. It means

$$\{\omega , a_r(\omega) \geq \tfrac{1}{2}\} \subseteq \{\omega , \mu_r(\omega) \geq \tfrac{1}{2}\} \qquad (3.7.)$$

Thus (3.5.) is the conclusion of (3.3.) , (3.6.) and (3.7.) .

Now, let us put

$$x(E_r)(\omega) = \begin{cases} \mu'(\omega) & f(\omega) \notin E_r \\ \mu(\omega) & f(\omega) \in E_r \end{cases} \qquad (3.8.)$$

for any $r \in Q$ and $x(R) = \mu$.

It can be shown that $x(E_r) \in M$ for any $r \in Q$.

Indeed, let $r \in Q$ and $\omega \in \Omega$. If $f(\omega) \in E_r$, then by (3.8.) $x(E_r)(\omega) = \mu(\omega)$. Since $\omega \in f^{-1}(E_r)$, by (3.5.) we have $\mu_r(\omega) \geq \frac{1}{2}$. These facts together with $\mu = \mu_r \vee \mu'_r$ mean that $x(E_r)(\omega) = \mu_r(\omega)$.

If $f(\omega) \notin E_r$, then by (3.8.) $x(E_r)(\omega) = \mu'(\omega)$. Simultaneously, since $\omega \notin f^{-1}(E_r)$, we get by (3.5.) $\mu_r(\omega) \leq \frac{1}{2}$. Due to the previous properties of μ we get $\mu(\omega) = \mu'_r(\omega)$ and further $x(E_r)(\omega) = \mu_r(\omega)$. So we have shown

$$x(E_r) = \mu_r \in M \quad \text{for any } r \in Q.$$

Now we are able to prove that any random variable f on the space $(\Omega, K(M))$ induces a fuzzy observable x of the fuzzy quantum space (Ω, M).

First, we give the following definition.

Definition 3.

A fuzzy set $u \in M$ is said to be a W-empty set, if $u \leq \frac{1}{2}$.

The set of all W-empty sets from M will be denoted by $W_0(M)$.

Theorem 2.

Let f be any random variable on the space $(\Omega, K(M))$. Then

(i) There exists a fuzzy observable x of the fuzzy quantum space (Ω, M) with the property (3.2.)

(ii) If y is any fuzzy observable of the fuzzy quantum space (Ω, M) satisfying (3.2.), then

$$x(E) \wedge y(E^c) \in W_0(M)$$

for any $E \in \mathcal{B}$.

Proof. (i) Let $E_r = (-\infty, r)$, $r \in Q$ and let us define $x(E_r)$ by (3.8.). By the previous part we have $x(E_r) = \mu_r \in M$ for any $r \in Q$. Since $\{E_r, r \in Q\}$ is a countable generator of the system $\mathcal{B}$, the function $x : \mathcal{B} \longrightarrow M$ is defined. The property (3.2.) of x follows from (3.5.), the fact $x(E_r) = \mu_r$ and the property of the system $\{E_r\}_{r \in Q}$, mainly $\mathcal{B} = \sigma(\{E_r, r \in Q\})$.

(ii) Let y is any fuzzy observable of the fuzzy quantum space (Ω, M) satisfying (3.2.) and let $E \in \mathcal{B}$. Then from the relations

$\{\omega, y(E^c)(\omega) > \frac{1}{2}\} \subseteq f^{-1}(E^c) \subseteq \{\omega, y(E^c)(\omega) \geq \frac{1}{2}\}$ and

$\{\omega, x(E)(\omega) > \frac{1}{2}\} \subseteq f^{-1}(E) \subseteq \{\omega, x(E)(\omega) \geq \frac{1}{2}\}$ we get

$\{\omega, x(E)(\omega) > \frac{1}{2}\} \cap \{\omega, y(E^c)(\omega) > \frac{1}{2}\} \subseteq f^{-1}(E) \cap f^{-1}(E^c) = \emptyset$.

Since $\{x(E) \wedge y(E^c) > \frac{1}{2}\} \subseteq \{x(E) > \frac{1}{2}\} \cap \{y(E^c) > \frac{1}{2}\}$

we have $\{x(E) \wedge y(E^c) > \frac{1}{2}\} = \emptyset$, what implies

$x(E) \wedge y(E^c) \leq \frac{1}{2}$, i.e. $x(E) \wedge y(E^c) \in W_0(M)$. ///

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