PAN-FUZZY INTEGRAL

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The concept of the pan-integral introduced in [1] which establishes the relation between the fuzzy integral and the classical integral is significant. In this paper, we shall introduce a family of functions which is called measurable function class and a kind of universal linear functional which is called pan-fuzzy integral, its some properties will be discussed. Pan-additive fuzzy measure can be regarded as a special example of the pan-fuzzy integral, and the real meaning of the pan-fuzzy integral defined in this paper is lastly given.

1. Preliminaries. It is assumed in this paper that $\bar{R}^+=[0,+\infty]$, $R^+=[0,+\infty)$, X is a nonempty set and the function we discussed takes value on $\bar{R}^+$ and defines in X.

Definition 1.1 Let " $\oplus$ " and " $\odot$ " are two kinds of binary operation on $\bar{R}^+$, satisfying the following conditions:

(1.1) $a \oplus b=b \oplus a$ (1.2) $(a \oplus b) \oplus c=a \oplus (b \oplus c)$

(1.3) $a_1 \le b_1$, $a_2 \le b_2 \Rightarrow a_1 \oplus a_2 \le b_1 \oplus b_2$

(1.4) $a \oplus o=a$

(1.5) If $\{a_n\}\subset\bar{R}^+$, $\{b_n\}\subset\bar{R}^+$ and $\lim\limits_{n\to\infty} b_n$, $\lim\limits_{n\to\infty} a_n$ exist then

$$\lim_{n\to\infty} (a_n \oplus b_n)=\lim_{n\to\infty} a_n \oplus \lim_{n\to\infty} b_n$$

(1.6) $a \odot b=b \odot a$ (1.7) $a \odot (b \odot c)=(a \odot b) \odot c$

(1.8) $(a \oplus b) \odot c=(a \odot c) \oplus (b \odot c)$

(1.9) $a_1 \le b_1$, $a_2 \le b_2 \Rightarrow a_1 \odot a_2 \le b_1 \odot b_2$

(1.10) $a \odot o=o$ (1.11) $a\neq o$, $b\neq o \Rightarrow a \odot b\neq o$

(1.12) Unit element $I \in \bar{R}^+$ exists, such that $I \odot a = a \odot I = a$

(1.13) If $\{a_n\} \subset \bar{R}^+$, $\{b_n\} \subset \bar{R}^+$, $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and are finite, then $\lim (a_n \odot b_n) = \lim_{n\to\infty} a_n \odot \lim_{n\to\infty} b_n$

(1.14) $\lim_{n\to\infty} \min(n, I) = I$

where a, b, c, a_i, $b_i \in \bar{R}^+$ (i=1, 2), "0" is number zero.

Then R is called exchange order-preserving semiring, denoted by $(\bar{R}^+, \oplus, \odot)$.

Example : $(\bar{R}^+, +, \cdot)$; $(\bar{R}^+, \vee, \cdot)$ and $(\bar{R}^+, \vee, \wedge)$ are exchange order-preserving semirings. where " + " and " $\cdot$ " are add operation and multiplication operation of the real number. $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$, $\forall a, b \in \bar{R}^+$. Their unit element are 1, 1, $+\infty$ respectively.

Definition 1.2 Let E be a subset of X, $\chi_E(x) = \begin{cases} I & x \in E \\ 0 & x \notin E \end{cases}$ is called characteristic function of E, where I is unit element of $(\bar{R}^+, \oplus, \odot)$.

In the following, we suppose that $\mathcal{L}$ is a family of functions, satisfying $\forall \varphi_1, \varphi_2 \in \mathcal{L}$, c_1, $c_2 \in \bar{R}^+$, such that

(1) $(c_1 \odot \varphi_1) \oplus (c_2 \odot \varphi_2) \in \mathcal{L}$

(2) $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$, $c_1 \varphi_1$, $(\varphi_1 - \varphi_2)^+$, $\varphi_1 \wedge I$, C_1, $I \in \mathcal{L}$

(3) If $\{f_n\} \subset \mathcal{L}$, $f_n \nearrow f$ or $f_n \searrow f$, then $f \in \mathcal{L}$.

Definition 1.3 Let $\mathcal{F}$ be a function family, such that for an arbitrary monotone function sequences $\{f_n\} \subset \mathcal{F}$, if $f(x) = \lim_{n\to\infty} f_n(x)$, then $f \in \mathcal{F}$. $\mathcal{F}$ is called a monotone class.

For every function family $\mathcal{F}$, the smallest monotone class containing $\mathcal{F}$ is called the monotone class generated by $\mathcal{F}$, denoted by $\mathfrak{B}(\mathcal{F})$, it is clear that $\mathcal{L} = \mathfrak{B}(\mathcal{L})$ and $\mathcal{L}$ is a monotone class.

Definition 1.4 $\forall \varphi \in \mathcal{L}$, if there is unique element in $(\bar{R}^+ \oplus, \odot)$ corresponding with φ, denoted by $\int \varphi$, satisfying the following conditions:

(4) $\forall \varphi_1, \varphi_2 \in \mathcal{L}$, $c_1 \geq 0, c_2 \geq 0$ then $\int[(c_1 \odot \varphi_1) \oplus (c_2 \odot \varphi_2)] = (c_1 \odot \int \varphi_1) \oplus (c_2 \odot \int \varphi_2)$

(5) $\varphi_1 \leq \varphi_2 \Rightarrow \int \varphi_1 \leq \int \varphi_2$

(6) $\{f_n\} \subset \mathcal{L}$, $f_n \nearrow f$ or $f_n \searrow f$, then $\int f_n \to \int f$

then $\int \varphi$ is called pan-fuzzy integral of φ on X.

2. Measurable functions and Measurable sets

Definition 2.1 A function f is called measurable function if $f \in \mathcal{L}$, if E is a subset of X and $\chi_E \in \mathcal{L}$, then E is called a measurable set. The all of the measurable sets is called a measurable set class generated by $\mathcal{L}$, denoted by $S(\mathcal{L})$.

Remark: In general, $\mathcal{B}(\mathcal{L})$ is called measurable class. As the special case of this paper, we suppose that $\mathcal{L}$ is a monotone class ($\mathcal{L} = \mathcal{B}(\mathcal{L})$).

In this section, we have the following results:

Proposition 2.1 If $f_n \in \mathcal{L}$, n=1, 2, ... , and let

(i) $g(x)=\sup(f_n(x))$ (ii) $h(x)=\inf(f_n(x))$

(iii) $\bar{f}(x)=\overline{\lim_n} f_n(x)$ (i) $\underline{f}(x)=\underline{\lim_n} f_n(x)$

then g, h, $\bar{f}$, $\underline{f}$ are measurable functions.

Proposition 2.2 : $S(\mathcal{L})$ is a σ- algebra.

Proposition 2.3 : A function f is measurable if and only if for every real number $a > 0$, the set $\{x \mid f(x) > a\}$ is measurable.

3. The pan-additive fuzzy measure defined by pan-fuzzy integral and the meaning of the pan-fuzzy integral

Definition 3.1 $(X, \mathcal{F}, u, \bar{R}^+, \oplus, \odot)$ is called universal space. where $\mathcal{F}$ is a σ- algebra of subsets of X, u is a fuzzy measure [1]

If u satisfies

$$u(E \cup F) = u(E) \oplus u(F)$$

whenever $E \in \mathcal{F}$, $F \in \mathcal{F}$ and $E \cap F = \phi$, then u is called a pan-additive fuzzy measure.

In this section, we have mainly the following results:

Proposition 3.1 u is pan-additive $\Rightarrow$ u is null-additive [2]

Theorem 3.1 Let $u(E)= \int \chi_E \qquad \forall E \in S(\mathcal{L})$.

then u is a pan-additive fuzzy measure on $S(\mathcal{L})$.

Theorem 3.2 Let $f \in \mathcal{L}$, i.e. , f is measurable, then $\int f=\int f du$.

where $\int f du$ is defined by [1] and u is the pan-additive measure.

We proof only theorem 3.2 .

[Proof] Deviding two cases:

(1) $f= \chi_F$, $F \in S(\mathcal{L})$, then $\int f=u(F) \overset{[1]}{=} \int \chi_F du= \int f du$

(2) Let f be a general measurable function, by proposition3.3, there is a sequence of functions

$$f_n = \bigoplus_{m=1}^{n\cdot 2^n} \left(\frac{m}{2^n} \odot \chi_{A_{n,m}} \right)$$

such that $f_n \nearrow f$. where $A_{n,m} = \left\{ x \,\middle|\, \frac{m}{2^n} < f(x) \leq \frac{m+1}{2^n} \right\}$. Hence

$$\begin{aligned} f &= \lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int \bigoplus_{m=1}^{n\cdot 2^n} \left(\frac{m}{2^n} \odot \chi_{A_{n,m}} \right) \\ &= \lim_{n\to\infty} \left[\bigoplus_{m=1}^{n\cdot 2^n} \left(\frac{m}{2^n} \odot \int \chi_{A_{n,m}} \right) \right] \\ &= \lim_{n\to\infty} \left[\bigoplus_{m=1}^{n\cdot 2^n} \left(\frac{m}{2^n} \odot u(A_{n,m}) \right) \right] \overset{[1]}{=} \int f du \end{aligned}$$

[Example] :

(i) Let $(\bar{R}^+, \oplus, \odot)$ be $(\bar{R}^+, +, \cdot)$, $\mathcal{L}$ be all of nonegative measurable functions then $\int f=(L)\int f du$, where $(L)\int f du$ is Lebesgue's integral.

(ii) Let $(\bar{R}^+, \oplus, \odot)$ be $(\bar{R}^+, \vee, \wedge)$, $\mathcal{L}$ be all of nonegative functions (measurable), then $\int f=(s)\int f du$, where u is the pan-addtive measure defined in theorem 3.1 , and $(s)\int f du$ is Sugeno's fuzzy integral [3] , it is defined as $(S)\int f du=\sup_{\alpha\in[0,\infty]}[\alpha \wedge u(\{x \mid f(x)\geq\alpha\})]$

Reference :

[1] Yang Qingji , The Pan-integral on The Fuzzy Measure Space, Fuzzy Mathematics (CHINA), 3 (1985), 107-114 .

[2] Wang Zhenyuan, Asymptotic Structural Characteristics of Fuzzy Measure and Their Application, Fuzzy Sets and Systems, 16 (1985), 277-290 .

[3] M. Sugeno, Theory of fuzzy integrals and its applications, Thesis, Tokyo Institute of Technology (1974).