g_λ-MEASURES AND CONDITIONAL g_λ-MEASURES ON MEASURE SPACES

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In this paper, we introduce the concepts of g_λ-measures, conditional g_λ-measures and λ-independence associated with a μ-density g on a measure space $(X, \mathcal{A}, \mu)$. Some useful results about them are obtained.

Keywords: g_λ-measure, conditional g_λ-measure, λ-indenpendence.

1. Introduction

In his thesis, M.Sugeno has introduced a class of λ-additive measures g_λ(called g_λ-measures throughout this paper) on a measurable space $(X, \mathcal{A})$ [1] . It has been pointed out that if X is a finite set then a g_λ-measure on $(X,P(X))$ is entirely determined by the parameter $\lambda\in(-1,\infty)$ and the numbers $g_i=g(\{x_i\})$ $(i=1,2...n)$ which are said to be the density of g_λ. M.Berres has generalized this fact to a measure space [3]. It has been shown that any g_λ-measure can be represented by a density g on a measure space.

In this paper, we present a alternate approach to g_λ-measures associated with a μ-density g on a measure space $(X, \mathcal{A}, \mu)$. We also propose the concepts of conditional g_λ-measures and λ-inde--pendence associated with a μ-density g on a measure space $(X, \mathcal{A}, \mu)$.Some useful results, such as the analoues of Bayes' formula, extension theorem of independent class and Borel-

-Cantelli's lemma, are obtained .

2. g_λ-Measures on Measure Spaces

We start with the following lemmas:

Lemma 2.1. Suppose $g: X \longrightarrow [0, 1]$ is a μ-integrable function on a measure space $(X, \mathcal{A}, \mu)$. Then $G(\lambda) = \int_X \log_{1+\lambda}(1+\lambda g)d\mu$ is a strictly increasing function on $(-1, 0)\cup(0, \infty)$ and

$$\lim_{\lambda \to 0} G(\lambda) = \int_X g d\mu , \qquad \lim_{\lambda \to \infty} G(\lambda) = \mu(\{0 < g \leq 1\}) ,$$

$$\lim_{\lambda \to -1^+} G(\lambda) = \mu(\{g = 1\})$$

Lemma 2.2. The Eq. $G(\lambda) = 1$ has a unique root $\lambda \in (-1, 0)\cup(0, \infty)$ which is positive (negative) when $\int_X g d\mu < 1$ ($\int_X g d\mu > 1$) iff μ and g satisfying

$$\mu(\{0 < g \leq 1\}) > 1 \qquad (2.1)$$

and

$$\mu(\{g = 1\}) < 1 \qquad (2.2)$$

We omit the proofs of lemma 2.1 and lemma 2.2 (cf.[3] ,[4]).

Theorem 2.3. Suppose $(X, \mathcal{A}, \mu)$ is a measure space and $g: X \longrightarrow [0, 1]$ is a μ-integrable function satisfying (2.1) and (2.2). If $\lambda \in (-1, 0)\cup(0, \infty)$ is the unique root of the Eq. $G(\lambda) = 1$, then $g_\lambda: \mathcal{A} \longrightarrow [0, 1]$ defined by

$$g_\lambda(\cdot) = \frac{1}{\lambda}\left[(1+\lambda)^{\int_{(\cdot)} \log_{1+\lambda}(1+\lambda g)d\mu} - 1\right] \qquad (2.3)$$

is a g_λ-measure on $(X, \mathcal{A})$.

Proof. Straightforward.

Remark: When $\int_X g d\mu = 1$ and g_λ is given by (2.3), then $\lim_{\lambda \to 0} g_\lambda(\cdot) = \int_{(\cdot)} g d\mu$ yields a probability measure g_0.

We name the function g satisfying (2.1) and (2.2) a μ-density and call the g_λ given by (2.3) the g_λ-measure associated with μ-density g on the measure space $(X, \mathcal{A}, \mu)$.

Obviously, if g_λ is a g_λ-measure associated with a μ-density g, then $g_\lambda^*(\cdot) = \log_{1+\lambda}(1+\lambda g_\lambda(\cdot)) = \int_{(\cdot)} \log_{1+\lambda}(1+\lambda g)d\mu$ is a probability measure. Moreover, there are the following properties of the g_λ-measures associated with μ-density g on $(X, \mathcal{A}, \mu)$:

Theorem 2.4. g_λ is a belief function iff $\int_X g d\mu \leqslant 1$ and it is a plausibility function iff $\int_X g d\mu \geqslant 1$.

By lemma 2.2, the proof of theorem 2.4 is obvious.

Theorem 2.5. Suppose $\{A_i\}_1^\infty$ are disjoint sets in $\mathcal{A}$. If $\int_{A_i} \log_{1+\lambda}(1+\lambda g)d\mu \neq 0$ for all i, then $\sum_{i=1}^\infty g_\lambda(A_i) >, =$ or $<$ $g_\lambda(\bigcup_{i=1}^\infty A_i)$ iff $\int_X g d\mu >, =$ or < 1, respectively.

Proof. The condition $\int_{A_i} \log_{1+\lambda}(1+\lambda g)d\mu \neq 0$ ensures $g_\lambda(A_i) > 0$. Since $\sum_{i=1}^\infty g_\lambda(A_i) >, =$ or $< g_\lambda(\bigcup_{i=1}^\infty A_i)$ iff $\lambda <, =$ or > 0, respec--tively when $g_\lambda(A_i) > 0$ for all i [5], we get the proof of theorem 2.5 from lemma 2.2 immediately.

3. Conditional g_λ-Measures and λ-Independence on Measure Spaces

Definition 3.1. Let g be a μ-density on a measure space $(X, \mathcal{A}, \mu)$. If $\int_X g d\mu \neq 1$ and $\int_A \log_{1+\lambda}(1+\lambda g)d\mu \neq 0$ $(A \in \mathcal{A})$, we define

$$g_\lambda(B|A) = \frac{1}{\lambda}\left[(1+\lambda)^{\int_{B|A} \log_{1+\lambda}(1+\lambda g)d\mu} - 1\right] \qquad (3.1)$$

and name $g_\lambda(\cdot|A): \mathcal{A} \longrightarrow [0, 1]$ the conditional g_λ-measures asso--ciated with μ-desity g given A. Where the paremeter λ is the unique root of the Eq. $G(\lambda) = 1$ and

$$\int_{B|A} \log_{1+\lambda}(1+\lambda g)d\mu = \int_{A \cap B} \log_{1+\lambda}(1+\lambda g)d\mu \Big/ \int_A \log_{1+\lambda}(1+\lambda g)d\mu$$

The conditional g_λ-measures associated with μ-density g given A have the following properties:

Theorem 3.2. $g_\lambda(\cdot|A)$ is a g_λ-measure on $(X,\mathcal{A})$ and

$$\log_{1+\lambda}(1+\lambda g_\lambda(B|A)) = \log_{1+\lambda}(1+\lambda g_\lambda(A\cap B))/\log_{1+\lambda}(1+\lambda g_\lambda(A))$$

for all $B\in\mathcal{A}$. Where g_λ is given by (2.3).

Theorem 3.3. Suppose $B\in\mathcal{A}$ and $\{A_i\}$ are disjoint sets in $\mathcal{A}$ satisfying $\bigcup_i A_i \supseteq B$. Then

$$\int_B \log_{1+\lambda}(1+\lambda g)d\mu = \sum_i \int_{B|A_i}\log_{1+\lambda}(1+\lambda g)d\mu\cdot\int_{A_i}\log_{1+\lambda}(1+\lambda g)d\mu$$

and

$$\int_{A_j|B}\log_{1+\lambda}(1+\lambda g)d\mu = \frac{\int_{B|A_j}\log_{1+\lambda}(1+\lambda g)d\mu\cdot\int_{A_j}\log_{1+\lambda}(1+\lambda g)d\mu}{\sum_i\int_{B|A_i}\log_{1+\lambda}(1+\lambda g)d\mu\cdot\int_{A_i}\log_{1+\lambda}(1+\lambda g)d\mu}$$

The later is the Bayes' -like formula. The proofs of above theorems are straightforward.

Definition 3.4. Let g be a μ-density on a measure space $(X,\mathcal{A},\mu)$, $\int_X g du \neq 1$ and λ be the unique root of the Eq. $G(\lambda) = 1$. Sets $A_1,A_2,\ldots A_n (n \geqslant 2)$ are said to be λ-independent associated with μ-density g if, for any $2\leqslant m\leqslant n$ and any $1\leqslant k_1<k_2<\ldots<k_m\leqslant n$,

$$\int_{\bigcap_{i=1}^m A_{k_i}}\log_{1+\lambda}(1+\lambda g)du = \prod_{i=1}^m\int_{A_{k_i}}\log_{1+\lambda}(1+\lambda g)d\mu \qquad (3.3)$$

Definition 3.5. The sets of the classes $\mathcal{D} = \{A_t\in\mathcal{A}, t\in T\}$ are said to be λ-independent associated with μ-density g if, for any non-empty finite subset $S\subseteq T$,

$$\int_{\bigcap_{s\in S} A_s}\log_{1+\lambda}(1+\lambda g)d\mu = \prod_{s\in S}\int_{A_s}\log_{1+\lambda}(1+\lambda g)d\mu \qquad (3.4)$$

A group of classes $\{\mathcal{C}_t : t\in T\}$ is said to be a λ-independent class associated with μ-density g, if the sets of the class $\mathcal{D} = \{A_t\in\mathcal{A} | A_t\in\mathcal{C}_t, t\in T\}$ are λ-independent associated with μ-density g.

On a similar plan we can achieve the analogous results of the λ-independence defined above with the probabilistic case. We only present two important theorems without their proofs here:

Theorem 3.6. (Extension theorem of λ-independent class) Suppose $\{\mathcal{C}_t, t\in T\}$ is a λ-independent class associated with μ-density g. If for every $t\in T$, $\mathcal{C}_t$ is closed under the finite intersections, then $\{\sigma(\mathcal{C}_t), t\in T\}$ is a λ-independent class associated with μ-den--sity g. Where $\sigma(\mathcal{C}_t)$ represents the σ-algebra generated by $\mathcal{C}_t$.

Theorem 3.7. (Borel-Cantelli like lemma) Let g be a μ-density on a measure space $(X, \mathcal{A}, u)$ and $\{A_n\}_1^\infty$ a seqence of sets in $\mathcal{A}$.

(1) If $\sum_{n=1}^{\infty}\int_{A_n}\log_{1+\lambda}(1+\lambda g)d\mu<\infty$, then $\int_{\overline{\text{limit}}_{n\to\infty} A_n}\log_{1+\lambda}(1+\lambda g)d\mu=0$;

(2) If $\{A_n, n\geq 1\}$ are λ-independent associated with μ-density g and $\int_{\overline{\text{limit}}_{n\to\infty} A_n}\log_{1+\lambda}(1+\lambda g)du=0$, then $\sum_{n=1}\int_{A_n}\log_{1+\lambda}(1+\lambda g)d\mu<\infty$.

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