The Least Upper Bound of Content for Realizable Matrices on Lattice [0, 1]

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In this paper, we shall solve the problem of the least upper bound of content for realizable matrices on the completely distributive lattice $([0, 1], \leq, \vee, \wedge)$.

1. Introduction

In reference [1], Liu Wangjin introduced the concept of a realizable fuzzy symmetric matrix and its content and obtained the result such that a fuzzy symmetric matrix $B_{2\times 2}$ is realizable iff $B \cdot B = B$. In the following, Che Yijiang [2] and Wang Mingxin [3] gave the necessary and sufficent condition for $B = (a_{ij})_{n\times n}$ to be realizable was that $b_{ii} \geq b_{ij} = b_{ji}$ $(i, j = 1, 2, \ldots, n)$. Yu Yandong [4] extended the problem of realizable fuzzy to L-fuzzy matrix and proved that the necessary and sufficient condition for realizable L-fuzzy matrix was still $b_{ii} \geq b_{ij} = b_{ji}$ $(i, j = 1, 2, \ldots, n)$, where L was any lattice. Wang Geping [5] discussed the problem of realizable L-fuzzy matrix on completely distributive lattice L and obtained some results on the estimation of $r_n(L)$ – the least upper bound of content for realizable matrices on L. Particularly, when $L = ([0, 1], \leq, \vee, \wedge)$, where '$\leq$' is the common order of real numbers, Wang [5] gave the following result: for $n \geq 4$, there holds

$$\max\{n, (n-1)^2/12\} \leq r_n(L) \leq (n-1)(n-2)/2 + 1.$$

The deep result was gotton by Zhao Duo in [6]: if L is the lattice mentioned above, B is a realizable matrix, then

$$\nu(B) \leq r(B) \leq [n^2/4],$$

where $r(B)$ is the content of B, $\nu(B)$ is the rank of B and $[n^2/4]$ is the integral part of $n^2/4$. From this conclution, we have

$$r_n(L) \leq [n^2/4].$$

In this article, we shall restrict $L = ([0, 1], \leq, \vee, \wedge)$

and solve this premble on this lattice. We get: for $n \leq 3$,

$$r_n(L) = n$$

and for $n \geq 4$,

$$r_n(L) = [n^2/4].$$

2. Prelimiaries

Let $L^{n\times m} = \{(a_{ij})_{n\times m};\ a_{ij} \in [0,\ 1],\ i = 1,\ 2, \ldots,\ n,\ j = 1,\ 2, \ldots,\ m\}$.

Definition 2.1 Let $A = (a_{ij}) \in L^{n\times p}$, $B = (b_{kj}) \in L^{p\times m}$. Note

$$A \cdot B = (c_{ij}) \in L^{n\times m},$$

where

$$c_{ij} = \vee_{k=1}^{p} (a_{ik} \wedge b_{kj}),\ i = 1,\ 2, \ldots,\ n,\ j = 1,\ 2, \ldots,\ m.$$

$A \cdot B$ is called the product of A, B.

Definition 2.2 Let $B = (b_{ij}) \in L^{n\times n}$. If there exists $A \in L^{n\times m}$ such that $B = A \cdot A'$, where A' is the thanpose of A, we call B is realizable and $A \cdot A'$ is one decomposition of B.

From [2,3,4], $B = (b_{ij}) \in L^{n\times n}$ is realizable iff

$$b_{ii} \geq b_{ij} = b_{ji}$$

for all $i,\ j = 1,\ 2, \ldots,\ n$.

Definition 2.3 Let $B \in L^{n\times n}$ be realizable. Note

$$r(B) = \min\{m;\ \text{there exists } A \in L^{n\times m} \text{ such that } A \cdot A' = B\}.$$

$r(B)$ is called the content of B.

Definition 2.4 The least upper bound of content for all realizable matrices $B \in L^{n\times n}$ is defined by

$$r_n(L) = \min\{\alpha;\ \text{for all realizable matrices } B \in L^{n\times n},\ r(B) \leq \alpha\}.$$

Since $L = ([0,\ 1],\ \leq,\ \vee,\ \wedge)$ in this paper, we note $r_n(L)$ by r_n simply.

3. On r_n

In this part, we shall prove $r_n = [n^2/4]$ if $n \geq 4$ and $r_n = n$ if $n \leq 3$.

Theorem 3.1 For all $n \geq 4$,

$$r_n = [n^2/4].$$

Proof. Let $B = (b_{ij}) \in L^{n\times n}$ (cf. [5]),

where

$$b_{ij} = \begin{cases} 1, & i = j \text{ or } i-j \text{ is odd,} \\ 0, & \text{otherwise} \end{cases}$$

Obviously, B is realizable.

Let $A = (a_{ij}) \in L^{n\times m}$ such that $A \cdot A' = B$. Since B is a 0-1 matrix(that is the elements of B are 0 or 1) and $([0, 1], \leq, \vee, \wedge)$ is a totally orderly lattice, we can suppose that A is a 0-1 matrix. We can prove: for every $j = 1, 2, \ldots, m$, the j-th column of A contains at most two 1s. If not, there exist $1 \leq i_1 < i_2 < i_3 \leq n$ such that $a_{i_1} = a_{i_2} = a_{i_3} = 1$, then

$$b_{i_1 i_2} = b_{i_1 i_3} = b_{i_2 i_3} = 1.$$

From $b_{i_1 i_2} = 1$ and $b_{i_1 i_3} = 1$, we obtain that i_1 is odd and i_2, i_3 are even or i_1 is even and i_2, i_3 are odd. But this contradicts with $b_{i_2 i_3} = 1$.

Let $b_{ij} = 1$, $i < j$. since every column of A contains at most two 1s, we can easily prove that A must have one column being p_j^i, where p_j^i is the column that the i-th and j-th elements are 1 and the remainders are 0.

Let $R = \{(i, j);\ 1 \leq i < j \leq n,\ b_{ij} = 1\}$, $\sigma = \bar{\bar{R}}$(the power of R).

If n is even, we have

$$\sigma = n^2/4 = [n^2/4];$$

if n is odd, we have

$$\sigma = (n-1)^2/4 = [n^2/4].$$

Therefore, for all $n \geq 4$,

$$r(B) \geq \sigma = [n^2/4].$$

Consequently,

$$r_n \geq [n^2/4].$$

Note: Let $C = (p_j^i)_{(i,j)\in R}$, we can verify that

$$B = C \cdot C'.$$

From [6], for every realizable matrix $B \in L^{n\times n}$, there holds

$$r(B) \leq [n^2/4],$$

(In part 4, we shall demonstrate the same result using different method)So,

$$r_n \leq [n^2/4].$$

Combining with $r_n \geq [n^2/4]$, we get: for $n \geq 4$,

$$r_n = [n^2/4]. \qquad \square$$

If $n = 1, 2, 3$, $r_n = n$ can be proved easily.

4. One decomposition of realizable matrix

In this part, we shall give one decomposition of realizable matrix $B \in L^{n\times n} (n \geq 4)$: $B = A\cdot A'$ such that $A \in L^{n\times[n^2/4]}$. This decomposition also demonstrates $r_n \leq [n^2/4]$ if $n \geq 4$.

The following result was obtained in [5].

If $B = (b_{ij}) \in L^{n\times n}$ is realizable, then there exists $\hat{B} = (\hat{b}_{ij}) \in L^{n\times n}$ such that

(1) $\hat{B}$ is obtained by means of interchanging rows and correspond columns of B. So, $B = C\hat{B}C'$, where C is an element matrix and $\det(C) = 1$;

(2) for every $i = 1, 2, \ldots, n-2$, $\hat{b}_{i,i+1}$ is the greatest element in $\{\hat{b}_{ij};\ j \geq i+1\}$;

(3) B is realizable and $r(\hat{B}) = r(B)$.

We can verify that if $\hat{B} = A\cdot A'$, then

$$B = (CA)\cdot(CA)\ .$$

Therefore, the realizable matrix problem can be reduced to the case of the special matrix mentioned above. So, in the following, we suppose that the realizable matrix satisfy the contion (2).

If $n \geq 4$, $B \in L^{n\times n}$ is realizable, we construct $A \in L^{n\times[n^2/4]}$ such that $B = A\cdot A'$ using mathematical induction.

$n = 4$, let $B = (b_{ij}) \in L^{4\times 4}$ is realizable and

$$A = \begin{bmatrix} b_{11} & b_{12} & b_{13} & 0 \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & b_{33} & b_{34} \\ b_{14} & 0 & 0 & b_{44} \end{bmatrix}$$

it is easy to show that $B = A\cdot A'$ and $A \in L^{4\times[4^2/4]}$.

Suppose if $n = k$ and $B \in L^{k\times k}$ is realizable, the there exists $A \in L^{k\times[k^2/4]}$ such that $B = A\cdot A'$.

Let $n = k+1$ and $B = (b_{ij}) \in L^{(k+1)\times(k+1)}$ be realizable. Note

$$B_1 = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k1} & b_{k2} & \cdots & b_{kk} \end{bmatrix}$$

then B_1 is realizable. From hypothesis, there exists $A_1 \in L^{k\times[k^2/4]}$ such $B_1 = A_1\cdot A_1'$.

Case 1. If k is even, let

$$A = \left[\begin{array}{c|cccccc} & b_{1,k+1} & & & & & \\ & b_{2,k+1} & & & & & \\ A_1 & & b_{3,k+1} & & & & \\ & & b_{4,k+1} & & & & \\ & & & \ddots & & & \\ & & & & b_{k-1,k+1} & & \\ & & & & b_{k,k+1} & & \\ \hline O & b_{k+1,k+1} & b_{k+1,k+1} & \cdots & b_{k+1,k+1} & & \end{array}\right]$$

The number of columns of A is $[k^2/4] + k/2 = [(k+1)^2/4]$.

Since $b_{12} \doteq b_{1,k+1}$, $b_{34} \doteq b_{3,k+1}, \ldots,$ $b_{k-1,k} \doteq b_{k-1,k+1}$, we have

$$b_{12} \doteq b_{1,k+1} \wedge b_{2,k+1},$$
$$b_{34} \doteq b_{3,k+1} \wedge b_{4,k+1},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$b_{k-1,k} \doteq b_{k-1,k+1} \wedge b_{k,k+1}.$$

From these results, we can verify that

$$B = A \cdot A'.$$

Case 2. If k is odd, let

$$A = \left[\begin{array}{c|ccccc} & b_{1,k+1} & & & & \\ & b_{2,k+1} & & & & \\ A_1 & & b_{3,k+1} & & & \\ & & b_{4,k+1} & \ddots & & \\ & & & & b_{k,k+1} & \\ \hline O & b_{k+1,k+1} & b_{k+1,k+1} & \cdots & b_{k+1,k+1} & \end{array}\right]$$

The number of columns of A is $[k^2/4] + (k+1)/2 = [(k+1)^2/4]$ and similary to case 1, we can obtain

$$B = A \cdot A',$$

the conclution follows.

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