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RES EARCH ON NONSTANDARD MATHEMATICS AND FUZZY SETS

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(i) Boolean Analysis

The nonstandard approach to fuzzy sets [1] is based on a Boolean generalization of Infinitesimal Analysis [2],[4],[6].

More precisely, we have the following framework:

For every point function $f^{\triangleright} : \Omega \to S$ there is a unique set function

$$f^{\triangleleft} : S \to \mathcal{P}(\Omega)$$

with $f^{\triangleleft}(s) := \Omega_s \equiv \{\omega \in \Omega : f^{\triangleright}(\omega) = s\}$

If $S^{\Omega} \equiv \mathrm{Hom}(\Omega, S)$ is the set of all point functions with domain Ω and co-domain S and

$$S[\mathcal{P}(\Omega)] := \{f \in \mathcal{P}(\Omega)^S : f(s_1) \cap f(s_2) = \emptyset, s_1 \neq s_2 \quad and \quad \cup_{s \in S} f(s) = \Omega\}$$

then $S[\mathcal{P}(\Omega)]$ is the Boolean power of S with respect to power Boolean algebra $\mathcal{P}(\Omega)$ and we have:

$$S^{\Omega} \cong S[\mathcal{P}(\Omega)]$$

that is the two sets of functions are isomorphic (1-1 and onto). If instead of the power Boolean algebra $\mathcal{P}(\Omega)$ we start with a σ-algebra of a probability space then we get the isomorphism of the set of all elementary random elements of S, denoted by $\mathcal{E}(S)$ and the general Boolean power S[$\mathbb{B}$], where $\mathbb{B}$ is the Boolean algebra induced by the given σ-algebra.

Using the theory of Boolean Powers, one can generalize the Robinson's Infinitesimal Analysis, using instead of the trivial probability space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), P_U)$ where P_U is the $0-1$ measure based on a free ultrafilter U to a general probability space $(\Omega, \mathcal{A}, P)$. This gives a special Boolean-valued model and a powerful Boolean Analysis, which is a direct generalization of Infinitesimal Analysis.

Actually we get to isomorphic models, the extensional and the intentional one. The extensional model is directly connected with Robinson's Infinitesimal Analysis which is based on Ultrapowers and intentional one is connected with Nelson's Internal Set Theory.(see [1]).

A last step that one might take, is a generalization of the above, towards topos theoretic characterizations of nonstandard extensions.

(ii) Boolean Fuzzy Sets

Using the above theory of Boolean Analysis, one can develop a theory of Fuzzy Sets which has all the ingredients of the usual Fuzzy Set theory, i.e. Extension Principle, cuts, etc. but in addition we have a transfer principle based on Łoś Theorem, and a rich logical structure.

It must be stressed that e.g. a function $f : \mathbb{R} \to \mathbb{B}$, with $f \in \mathbb{R}[\mathbb{B}]$ represents a $\mathbb{B}$-fuzzy number and not a $\mathbb{B}$-fuzzy subset of $\mathbb{R}$ and this is due to the fact that $f \in \mathbb{R}[\mathbb{B}]$. A $\mathbb{B}$-fuzzy subset of $\mathbb{R}$ is of the following form:

$$A(\cdot) : \mathcal{P}(\mathbb{R}) \to \mathbb{B}$$

and $A(\cdot) \in \mathcal{P}(\mathbb{R})[\mathbb{B}]$, etc.

Due to the above property a $\mathbb{B}$-fuzzy set and its membership function defined as :

$$\mu_A(\cdot) : \mathbb{R}[\mathbb{B}] \to \mathbb{B}$$

defined by $\mu_A(f) \equiv \|f \in A\| := \bigvee_{x \in y \subseteq \mathbb{R}} f(x) \wedge A(y)$ are different. The membership function assigns degrees of membership, not only , to standard reals but also to nonstandard. However the values on standard values determine completely the membership function. The above approach gives a powerful theory of $\mathbb{B}$-fuzzy sets and stochastic analysis, as a nonstandard real analysis. For technical details see [2],[3],[4], [1]. One should also note, that in this theory, the real fuzzy sets are the external ones. Loeb measures can be also transferred to this case, giving a powerful theory of fuzzy and stochastic measures as a nonstandard extension of real analysis.

Choosing various Boolean algebras we get various theories without any additional work. For example, if we choose $\mathbb{B}$ to be a Boolean algebras of projections on a Hilbert space we get Quantum Probability Theory, or if we choose a Boolean algebra coming from fuzzy partitions and soft probability spaces [7], we get a theory of fuzzy random elements and fuzzy stochastic spaces [5].

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