DECISION PROCEDURES WITH FUZZY BINARY RELATIONS

Leonid .M. Kitainik,

Computing center of the USSR Academy of sciences,

40, Vavilov street, 117333, Moscow, USSR

The paper contains brief review of a theory of **fuzzy decision procedures** with fuzzy binary relations.

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This paper is devoted to decision-making with fuzzy binary relations (**FRs**), more specially – to research of certain class of most convenient choice rules. General approach to fuzzy decision procedures with **FRs**, outlined below, resulted from the analysis of various decision rules with binary relations, used in both crisp and fuzzy theories (see Bezdek, Spillman and Spillman, 1978; Dubois and Prade, 1980, II.3, IV.3; Orlovsky, 1978; Roubens, 1989; Scwartz, 1986; Volsky, 1988 and many others). A review of the theory, developed in (Kitainik, 1986, 1987, 1988, 1989, 1990), containing most important concepts and results, is presented.

Let **X** be the finite set of initial crisp alternatives. A nonfuzzy binary relation on **X** is assosiated with oriented graph. In classical axiomatic choice theory, the "graphodominant" choice function, that is, the rule, selecting all ordinary non-dominated nodes of a graph (**CND** rule), is the basic one. The majority of fuzzy decision models also refer to one or another notion of "non-dominated alternatives" (**FND** – see Roubens, 1989). However, in crisp case the **CND** rule often results in empty choice. By this and some other reasons, various "non-classical" choice rules are widely used (Von Neumann – Morgenstern solution – **NMS**, intrinsic and external stability, **GOCHA** and **GETCHA** rules – see Scwartz, 1986, and so on). The diversity of the rules raises many problems; three of them are considered below: **systematization, comparative estimate** and **contensiveness**. It occured that systematization can be done within crisp theory; with the remaining two problems, fuzzy consideration is essential.

Systematization of choice rules (Kitainik, 1987; 1988; 1989; 1990). Most of the "invariant" choice rules, including the above-mentioned representatives, turned out to be "positive combinations" of three **basic dichotomies**, namely, **"R-non-domination"** Δ_1 (the principal concept of **GOCHA** rule), **intrinsic stability** Δ_2 and **external stability** Δ_3 . Basic rules are easily expressed in the similar algebraic manner by means of **composition law** $\odot$ and **inclusion** $\le$. Formally, a subset of alternatives $Z \le X$ satisfies Δ_i with given **FR R** , iff

$$\Delta_1 : R \odot \bar{Z} \le \bar{Z} ; \quad \Delta_2 : R \odot Z \le \bar{Z} ; \quad \Delta_3 : R \odot Z \ge \bar{Z} .$$

The study of composition laws shows that, in certain natural algebraic meaning, only two of them are worth notice: the fundamental boolean product $\square$ and the dual law $|\bar{\square}$ with

$R\ |\bar{\square}\ Z = \overline{\bar{R} \square Z}$. More specially, the mappings $Z \longrightarrow R \square Z$ represent all endomorphisms of semilattice $\mathcal{P}_{\cup}$ of all subsets of X with respect to union, and the family of mappings $Z \longrightarrow R\ |\bar{\square}\ Z$ exhausts all homomorphisms of $\mathcal{P}_{\cup}$ into the dual semilattice $\mathcal{P}_{\cap}$. "Ordinary" R-non-domination, intrinsic and external stability are expressed by $\Delta_1(\square)$-$\Delta_3(\square)$, whereas "**R-domination**" (the point of **GETCHA** rule) is merely a version of external stability $\Delta_3(|\bar{\square})$, based on composition law $|\bar{\square}$.

The next step is to consider meaningful combinations of basic dichotomies - all non-decreasing boolean polynomials, depending on Δ_1-Δ_3 .This construction results in a free distributive lattice (isomorphic to D_{18}) of **dichotomous choice rules**, containing 18 elements. Thus, **CND** concept corresponds to $\Delta_1 \wedge \Delta_2(\square)$, **NMS** - to $\Delta_2 \wedge \Delta_3(\square)$, "game-theoretical kernel" - to $\Delta_1 \wedge \Delta_2 \wedge \Delta_3(\square)$ etc. Other interesting choice rules, unknown in literature, can be also found in both $\square$- and $|\bar{\square}$-based families.

Note. All introduced choice rules generally result in **multifold** choice (for further discussion and axiomatics of multifold choice, see Bondareva, 1988, Kitainik, 1987).

Fuzzy decision procedures with FRs. Contensiveness criteria (Kitainik, 1987, 1988). To work out the tools for investigating consistency of choice rules in both crisp and fuzzy environment, one needs more general notions of **fuzzy decision procedure** and of its **contensiveness.**

A **fuzzy decision procedure** (FDP) with **FR** is defined as a mapping $\mathbf{p}: \mathcal{F}(X^2) \longrightarrow \mathcal{F}^{(2)}(X)$ of the set of all **FR**s into the set of all fuzzy level two subsets of a support **X.** The idea of introducing **FDP**s is that with **fuzzy** relation the choice rule must also be fuzzified, providing appropriate answer to the question "to what extent an arbitrary fuzzy subset satisfies choice conditions?". Given a **FDP p** and a **FR** $R \in \mathcal{F}(X^2)$,the value $\mu_{p(R)}(a)$ of membership function answers the question, being interpreted as a degree of conformance of a version of "a priori" preferences $\mathbf{a} \in \mathcal{F}(X)$ with the notion of optimality, represented by **p**, when analysing a **FR R** . As usually, Bellman-Zadeh maximal decision principle yields optimal solution

$$D(p,R,\mathcal{Y}) = \mu^{-1}_{p(R)}\ (\mu^{*}_{p(R)}\ (\mathcal{Y}))\quad (\ \mu^{*}_{p(R)}(\mathcal{Y}) = \bigvee_{a \in \mathcal{Y}} \mu_{p(R)}(a)\ ) -$$

an ordinary set of fuzzy subsets, best fitting the pair <procedure, **FR**> - a straightforward analogue of crisp "multifold" choice (here, an **environment** $\mathcal{Y} \subseteq \mathcal{F}(x)$ is a preliminarily selected domain of "admissible fuzzy preferences"; in **universal environment** $\mathcal{Y} = \mathcal{F}(x)$ the symbol $\mathcal{Y}$ in designations is omitted).

In these terms, **contensiveness** of a pair <procedure, relation> is identified with certain **coherence** of optimal solution, inducing a non-trivial crisp choice, being thought of as ultimate end of the procedure. Two forms of contensiveness criteria proved to be most adequate:

dichotomous contensiveness (**DC**) means roughly that all fuzzy subsets, contained in optimal solution, have guarranteed distinguishing power with respect to original alternatives in **X**; in such case, solution must contain neither constants nor "constant-like" sequences.

ranking contensiveness (**RC**) is a more rigid property, requiring that certain crisp partition of the support should be uniformly ranked by all representatives in optimal solution (see Kitainik, 1988).

The contensiveness of a procedure **p** in certain class of **FR**s $\mathcal{R}$ is defined by means of "$\exists$-convolution": a pair <**p**,**R**> is expected to be contensive with appropriate $R \in \mathcal{R}$.

General axiomatic theory of fuzzy decision procedures is, for the present, far from building. However, we have enough candidates – all the 36 $\square$- and $|\bar{\square}$-based dichotomous procedures – to verify the efficiency of former definitions.

Fuzzy dichotomous decision procedures . Fuzzy inclusions (Kitainik, 1986; 1987). The formal transference of the concept of dichotomous procedures to fuzzy case is easy: boolean composition turns into $\vee$-$\wedge$ (more generally, to $\vee$-$*$) composition; the dual law $|\bar{\square}$ is defined exactly as in crisp case – $R \,|\bar{\square}\, a = \overline{\bar{R} \,\square\, a}$; ordinary inclusion $\subseteq$ is to be changed for the fuzzy one, so that the whole family now depends on the two structural parameters:

$$\Pi_\Delta(\odot,\mathbf{inc}) = \{\text{non-decreasing } \vee\text{-}\wedge \text{-polynomials with three variables } \Delta_1(\odot,\mathbf{inc}),\ \Delta_2(\odot,\mathbf{inc})\ \Delta_3(\odot,\mathbf{inc}) \};$$

$$\mu_{\Delta_1(R)}(a) = \mu_{\mathbf{inc}}(R\odot\bar{a} ,\bar{a}) \ ; \quad \mu_{\Delta_2(R)}(a) = \mu_{\mathbf{inc}}(R\odot a ,\bar{a}) \ ;$$

$\mu_{\Delta_3(R)}(a) = \mu_{\mathbf{inc}}(\bar{a}, R\odot a)$ (more precisely, $\mu_{\Delta_i(\odot,\mathbf{inc},R)}$; the only requirement to **FR** is antireflexivity – $\mu_R(x,x)\equiv 0$).

However, the selection of fuzzy inclusion is not quite routine. There exist several dozens of constructions, both of empiric and of speculative nature. To this end, an axiomatic theory of **fuzzy inclusions** was developed in Kitainik (1986), differing from (Baldwin, Pilsworth, 1980) approach in rather "algebraic" than "logical" motivations.

With this theory, the set of all fuzzy inclusions is modelled ("realized") by the set of all non-increasing functions on the triangle $T = \{ (\alpha,\beta) \in [0;1]^2 \mid \alpha \geq \beta \}$

with fixed zero/unit values in the angles (such a function can be viewed as "non-assosiative **t**-conorm"). The properties of inclusions depend on the behavior of the corresponding functions at certain subtriangles, and continuity of inclusion is equivalent to continuity of the function. The characteristic elements of the family turned out two well-known inclusions: **discontinuous** L.Zadeh' inclusion $\leq$ - the only reflexive, transitive and antisymmetric representative; **continuous** "Kleene - Dienes implication" I_5 - the only "linear model". Among other results of the theory, the conflict between algebraic and topological properties is of interest: thus, there exists no **continuous** fuzzy inclusion, being both **reflexive** and **transitive.**

Contensive and trivial fuzzy dichotomous procedures

(Kitainik, 1987, 1988, 1989). With any composition law $\odot \in \{ \square , |\bar{\square} \}$ and fuzzy inclusion **inc** $\in \{ \leq , I_5 \}$, at least sixteen of the eighteen procedures in $\Pi_\Delta(\odot,\mathbf{inc})$ turned to be both dichotomously and ranking trivial in universal environment - that is, **not contensive with any FR.** The remaining two procedures are fuzzy versions of **NMS** $(\Delta_2 \wedge \Delta_3)$ and of **kernel** $(\Delta_1 \wedge \Delta_2 \wedge \Delta_3)$. When based on composition law $\square$ and on $\leq$ or I_5 as fuzzy inclusions, both procedures are dichotomously and ranking contensive.

Thus, application of fuzzy contensiveness concept considerably reduces the number of meaningful procedures in **fuzzy** case. The same argumentation is valid for **crisp** relations as well, if one admits extended scale of preferences for alternatives - [0,1] instead of $\{0,1\}$.

Description of optimal solutions (examples; for details, see Kitainik, 1987, 1988, 1989). To construct optimal solution of fuzzy **NMS** $p = \Delta_2 \wedge \Delta_3(\square, I_5)$, one must find strict median cut $R_{>1/2}$ of the **FR**, and select the subset $\mathcal{K}^*$ of the best fitting **NMS'** of this **crisp** relation (due to Bellman - Zadeh principle, when applied to **p(R)**) . In these terms, **D(p,R)** is described as a family of similar **interval fuzzy dichotomies:**

$$D(p,R) = \bigcup_{K \in K^*} [\mu^* \chi_K , \chi_K \vee \bar{\mu}^* \chi_{\bar{K}}] \Leftrightarrow \{[\mu^*,1]/K + [0, \bar{\mu}^*]/\bar{K}\}$$

with $\mu^* = \mu^*_{p(R)} > 1/2$. Guarranteed resolvability of optimal solution is $\mu^* - \bar{\mu}^* > 0$, and any $K \in K^*$ represents the resulting crisp choice, being preferred to $\bar{K}$ as $[\mu^*,1]$ to $[0, \bar{\mu}^*]$.

On the contrary, the structural analogue of **FND** procedure $p = \Delta_1 \wedge \Delta_2(\square, I_5)$ yields optimal solution $D(p,R) = \mathcal{F}(X) \cap \chi_{M_{>0}}$, $M_{>0}$ being the **CND** of $R_{>0}$. Hence, no confident preference of the crisp choice $M_{>0}$ can be found (e.g., with any $\varepsilon>0$, $\varepsilon\circ\chi_{M_{>0}} \in D(p,R)$). So, not the **potential emptyness** of "grapho-dominant choice", but its **actual triviality** is one more reason to carefully apply (or, maybe, avoid) this procedure.

Another version of fuzzy **NMS**, based on L.Zadeh' inclusion $\le$, $p = \Delta_2 \wedge \Delta_3(\square,\le)$, is generally less available than I_5-procedure; however, it often leads to proper **fuzzy interval ranking** by means of two stable points of "unity orbit" $\Omega_R = \{\varkappa_R^k(1)\}$ of the mapping $\varkappa_R: \mathcal{F}(X) \longrightarrow \mathcal{F}(X)$, $\varkappa_R(a) = \overline{R \square a}$. A $\le$-based **FNMS** throws new light on the nature of "conventional **FND**", $\mu_{FND(R)}(x) = 1 - \max_{y \in X\setminus\{x\}} \mu_R(y,x) \ (x \in X)$, widely used in various fuzzy decision models. **FND(R)** , being the first element in Ω_R, almost never belongs to optimal solution, even when **FR** is fuzzy ordering. To obtain **contensive** optimal solution, one must apply "$\varkappa_R$-construction" not once (**FND** case), but several times (2 times - not 1 ! - will be enough for transitive, 2∘card(X)-1 times - for arbitrary **FR**).

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