The Initial Value Problems of Fuzzy Differential Equations

Yunqing Wang
Department of Mathematics
Hebei Teacher's University
Shijiazhuang, Hebei, P.R.China

ABSTRACT Fuzzy Differential Equations (FDE) were introduced in 1987 in [1] , and used in some mathematical models in 1989 in [2]. Although the method of finding solution was given in [1], the problem of existness and uniqueness of solution has not been solved. This paper constructs a fuzzy function space (a complete metric space), and studies one type of fuzzy integral equations. Finally, the existness and uniqueness of solution of the initial value problem of FDE are proved.

KEY WORDS Differential of Fuzzy Functions, Integral of Fuzzy Functions, Fuzzy Function Space, Fuzzy Integral Equations, Fuzzy Differential Equations.

1 Fuzzy Functions Space

Definition 1.1 A fuzzy number is a fuzzy set $m : R \to I = [0,1]$ with the properties:

1. m is upper semicontinuous,
2. $m(x) = 0$, outside of some interval $[c, d]$,
3. there are real numbers a and b, $c \le a \le b \le d$ such that m is increasing on $[c, a]$, decreasing on $[b, d]$ and $m(x) = 1$ for each $x \in [a, b]$.

We use F to denote the family of fuzzy numbers. A real number a is a special fuzzy number as follows:

$$m_a = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

We can identify a fuzzy number m with parameterized triples

$$m = (a(r),\ b(r)), \qquad r \in I \qquad \text{............... (1.1)}$$

where

$$[a(r),\ b(r)] = \begin{cases} \{x \mid m(x) \ge r\} & \text{if } 0 < r \le 1 \\ cl(supp\, m) & \text{if } r = 0 \end{cases}$$

The following facts are clear:

1. $a(r)$ is a bounded increasing function, $b(r)$ is a bounded decreasing function,
2. $a(1) \le b(1)$,
3. for $0 < p \le 1$, $a(r)$ and $b(r)$ are left-hand continuous at p; $a(r)$ and $b(r)$ are right-hand continuous at $r = 0$.

If $a(r)$ and $b(r)$ satisfy the conditions 1, 2 and 3, then

$$m : R \to I \quad \text{defined by} \quad m(x) = sup\{r \mid a(r) \le b(r)\}$$

is a fuzzy number with parameterization given by (1.1).

Let $V = \{(a(r),\ b(r)) \mid r \in I,\ a(r)$ and $b(r)$ are bounded functions$\}$. The addition '+', scalar multiplication '$\cdot$' and metric 'D' are defined in V as follows

$$(a(r),\ b(r)) + (c(r),\ d(r)) = (a(r) + c(r),\ b(r) + d(r)) \qquad \text{............... (1.2)}$$

$$k \cdot (a(r),\ b(r)) = (k \cdot a(r),\ k \cdot b(r)) \qquad \text{............... (1.3)}$$

$$D[(a(r),\ b(r)),\ (c(r),\ d(r))] = sup_{r\in I}\ max\{|a(r) - c(r)|,\ |b(r) - d(r)|\} \qquad \text{............... (1.4)}$$

It is also clear that V is a linear topological space and a complete metric space.

Definition 1.2 A function $f : X \to F$ is said to be fuzzy function, where $X \subset R$.

The limits and derivatives of fuzzy functions can be defined as real functions similarly.

If $f(x) = (a(r,x),\ b(r,x))$ is differentiable, we have

$$f'(x) = (\frac{\partial a(r,x)}{\partial x},\ \frac{\partial b(r,x)}{\partial x})$$

Let $S = \{f \mid f : X \to F$ is a continuous fuzzy function$\}$. Defining metric in S

$$D_2(f_1,\ f_2) = sup_{x\in X}\ D(f_1(x),\ f_2(x))$$

We call S to be a fuzzy function space on X.

Theorem 1.1 Fuzzy function space S is a complete metric space.

Proof. Let $f_n(x) = (a_n(r,x),\ b_n(r,x)) \quad n = 1,\ 2,\ ...$ be a Cauchy sequence in S. Then

$$\begin{aligned}&\lim_{m,n\to\infty} D_2(f_m,\ f_n)\\ &= \lim_{m,n\to\infty} sup_{x\in X} D(f_m(x),\ f_n(x))\\ &= \lim_{m,n\to\infty} sup_{x\in X} sup_{r\in I}\ max\{|a_n(r,x) - a_m(r,x)|,\ |b_n(r,x) - b_m(r,x)|\}\\ &= 0.\end{aligned}$$

Hence,

$$\lim_{m,n\to\infty} |a_n(r,x) - a_m(r,x)| = 0, \qquad \text{uniform on } r, x.$$

$$\lim_{m,n\to\infty} |b_n(r,x) - b_m(r,x)| = 0, \qquad \text{uniform on } r, x.$$

Moreover, there exist real functions $a(r,x),\ b(r,x)$ such that

$$a_n(r,x) \to a(r,x), \qquad \text{uniform on } r, x \qquad \text{............... (1.5)}$$

$$b_n(r,x) \to b(r,x), \qquad \text{uniform on } r, x \qquad \text{............... (1.6)}$$

where $a(r,x),\ b(r,x)$ are bounded on r and continuous on x.

Now we prove that $f(x) = (a(r,x),\ b(r,x))$ is a fuzzy function. Since $a_n(r,x)$ and $b_n(r,x)$ are fuzzy functions, for any $x \in X$, $a_n(r,x)$ and $b_n(r,x)$ satisfy conditions 1, 2 and 3. By (1.5) and (1.6), we know that $a(r,x)$ and $b(r,x)$ satisfy 1, 2 and 3 too. Therefore, $f(x) = (a(r,x),\ b(r,x))$ is a fuzzy function, and $D_2(f_n, f) \to 0$, when $n \to \infty$. #

2 A Type of Fuzzy Integral Equations (FIE)

Definition 2.1 Suppose that $f(x)$ is a fuzzy function. If there exists a fuzzy function $F(x)$ such that $F'(x) = f(x)$, $F(x)$ is called a antiderivative of $f(x)$. $F(x) + m$ is called indefinite integral of $f(x)$, where m is a arbitrary fuzzy number. Denote $\int f(x)dx = F(x) + m$.

Definition 2.2 Let $f(x) = (a(r,x),\ b(r,x))$ be a fuzzy function on $[\alpha, \beta]$. If

1. $\int_\alpha^\beta a(r,x)dx$ and $\int_\alpha^\beta b(r,x)dx$ exist simultaneously,
2. $\int_\alpha^\beta a(r,x)dx$ and $\int_\alpha^\beta b(r,x)dx$ is a fuzzy number.

we say $f(x)$ is integrable on $[\alpha, \beta]$. Moreover,

$$\int_\alpha^\beta f(x)dx = (\int_\alpha^\beta a(r,x)dx,\ \int_\alpha^\beta b(r,x)dx)$$

is called the definite integral of $f(x)$ on $[\alpha, \beta]$.

The following two lemmas can be proved easily and their proofs are omitted.

Lemma 2.1 Suppose that $f(x) = (a(r,x),\ b(r,x))$ have antiderivatives. Then,
$F(x) = (\int a(r,x)dx,\ \int b(r,x)dx)$ is an antiderivative of $f(x)$.

Lemma 2.2 Let $f(x)$ is a fuzzy function on $X \subset R$. If the definite integral $\int_{x_0}^x f(x)dx$ exists for $x_0 \in X$ and any $x \in X(x_0 \leq x)$, then $F(x) = \int_{x_0}^x f(x)dx$ is a fuzzy function and $F'(x) = f(x)$.

Theorem 2.1 Suppose that

1. $g(x) = (c(r, x),\ d(r, x))$ is a continuous fuzzy function on $[x_0,\ x_1]$,

2. $K(x, y)$ is a real continuous function on Δ:
$$\Delta = \{(x, y) \mid x_0 \le x \le x_1, \quad x_0 \le y \le x\}$$
and there is a constant $L > 0$, such that $0 \le K(x, y) \le L$.

Then for any $\lambda \ge 0$, Volterra form integral equation
$$g(x) + \lambda \int_{x_0}^{x} K(x, t) f(t) dt \qquad \text{............... (2.1)}$$
have a unique continuous fuzzy function solution $f(x)$ on $[x_0, x_1]$.

Proof. First, we know easily that if $f(x) = (a(r, x),\ b(r, x))$ is a continuous fuzzy function on $[x_0,\ x_1]$, then $g(x) + \lambda \int_{x_0}^{x} K(x, t) f(t) dt$ is a continuous fuzzy function on $[x_0,\ x_1]$.

Let $S = \{f \mid f \text{ is a continuous fuzzy function on } [x_0,\ x_1]\}$. Make a mapping $M : S \to S$ as follows:
$$Mf(x) = g(x) + \lambda \int_{x_0}^{x} K(x, t) f(t) dt$$
we will show that M is a compressed mapping with the metric D_2. For any $f_1(x),\ f_2(x) \in V$, denote:
$$f_i(x) = (a_i(r, x),\ b_i(r, x)), \qquad i = 1, 2.$$
then $\quad Mf_i(x) = (g_a(r, x) + \lambda \int_{x_0}^{x} K(x, t) a_i(r, t) dt,\ g_b(r, x) + \lambda \int_{x_0}^{x} K(r, t) b_i(r, t) dt)$

since $\quad D_2(f_1, f_2) = sup_{x \in [x_0, x_1]} D(f_1(x),\ f_2(x)) \quad$ and
$$D(f_1(x), f_2(x)) = sup_{r \in I}\ max\{|a_1(r, x) - a_2(r, x)|,\ |b_1(r, x) - b_2(r, x)|\}$$
We have $\quad D(Mf_1(x),\ Mf_2(x))$
$$\begin{aligned}
&= sup_{r \in I}\ max\{|\lambda \textstyle\int_{x_0}^{x} K(x, t) a_1(r, t) dt - \lambda \int_{x_0}^{x} K(x, t) a_2(r, t) dt|,\\
&\qquad |\lambda \textstyle\int_{x_0}^{x} K(x, t) b_1(r, t) dt - \lambda \int_{x_0}^{x} K(x, t) b_2(r, t) dt|\}\\
&\le sup_{r \in I} \lambda L\ max\{\textstyle\int_{x_0}^{x} |a_1(r, t) - a_2(r, t)| dt,\ \int_{x_0}^{x} |b_1(r, t) - b_2(r, t)| dt\}\\
&\le \lambda L(x - x_0) sup_{r \in I}\ max\{max_{t \in [x_0, x]} |a_1(r, t) - a_2(r, t)|,\ max_{t \in [x_0, x]} |b_1(r, t) - b_2(r, t)|\}\\
&= \lambda L(x - x_0) sup_{r \in I} max_{t \in [x_0, x]}\ max\{|a_1(r, t) - a_2(r, t)|,\ |b_1(r, t) - b_2(r, t)|\}\\
&= \lambda L(x - x_0) max_{t \in [x_0, x]} sup_{r \in I}\ max\{|a_1(r, t) - a_2(r, t)|,\ |b_1(r, t) - b_2(r, t)|\}\\
&= \lambda L(x - x_0) max_{t \in [x_0, x]} D(f_1(t),\ f_2(t))
\end{aligned}$$
Hence
$$\begin{aligned}
D_2(Mf_1,\ Mf_2) &= sup_{x \in [x_0, x_1]} D(Mf_1(x),\ Mf_2(x))\\
&\le \lambda L sup_{x \in [x_0, x_1]} (x - x_0) max_{t \in [x_0, x]} D(f_1(t),\ f_2(t))\\
&\le \lambda L(x_1 - x_0) sup_{x \in [x_0, x_1]} D(f_1(x),\ f_2(x)) = \lambda L(x_1 - x_0) D_2(f_1, f_2)
\end{aligned}$$
So
$$D_2(Mf_1,\ Mf_2) = \lambda L(x_1 - x_0) D_2(f_1, f_2) \qquad \text{............... (2.2)}$$
Now we prove
$$D_2(M^n f_1,\ M^n f_2) \le \frac{\lambda^n L^n (x_1 - x_0)^n}{n!} D_2(f_1, f_2) \qquad \text{............... (2.3)}$$
with mathematical induction.

When $n = 1$, (2.3) is proved. Suppose that for natural number n (2.3) is true, then for natural number $n + 1$,
$$\begin{aligned}
&D_2(M^{n+1} f_1,\ M^{n+1} f_2) = sup_{x \in [x_0, x_1]} D(M^{n+1} f_1(x),\ M^{n+1} f_2(x))\\
&= sup_{x \in [x_0, x_1]} sup_{r \in I}\ max\{|\lambda \textstyle\int_{x_0}^{x} K(x, t)[M^n a_1(r, t) - M^n a_2(r, t)] dt|,\\
&\qquad |\lambda \textstyle\int_{x_0}^{x} K(x, t)[M^n b_1(r, t) - M^n b_2(r, t)] dt|\}\\
&\le sup_{x \in [x_0, x_1]} \lambda L sup_{r \in I}\ max\{\textstyle\int_{x_0}^{x} |M^n a_1(r, t) - M^n a_2(r, t)| dt,\ \int_{x_0}^{x} |M^n b_1(r, t) - M^n b_2(r, t)| dt\}\\
&\le sup_{x \in [x_0, x_1]} \lambda L sup_{r \in I}\ max\{\textstyle\int_{x_0}^{x} D(M^n f_1(t), M^n f_2(t)) dt,\ \int_{x_0}^{x} D(M^n f_1(t), M^n f_2(t)) dt\}\\
&= sup_{x \in [x_0, x_1]} \lambda L \textstyle\int_{x_0}^{x} sup_{\bar{t} \in [x_0, t]} D(M^n f_1(\bar{t}),\ M^n f_2(\bar{t})) dt\\
&\le sup_{x \in [x_0, x_1]} \lambda L \textstyle\int_{x_0}^{x} \frac{\lambda^n L^n (t - x_0)^n}{n!} D_2(f_1, f_2) dt\\
&\qquad \le \lambda L sup_{x \in [x_0, x_1]} \lambda^n L^n \tfrac{1}{n!} D_2(f_1, f_2) \textstyle\int_{x_0}^{x} (t - x_0)^n dt\\
&\le \frac{\lambda^{n+1} L^{n+1}}{n!} D_2(f_1, f_2) \textstyle\int_{x_0}^{x_1} (t - x_0)^n dt = \frac{\lambda^{n+1} L^{n+1} (x_1 - x_0)^{n+1}}{(n+1)!} D_2(f_1, f_2)
\end{aligned}$$
Since
$$\lim_{n \to \infty} \frac{\lambda^n L^n (x_1 - x_0)^n}{n!} = 0$$
there exists a natural number N, such that for $n = N$
$$h = \frac{\lambda^N L^N (x_1 - x_0)^N}{N!} < 1$$

That means $D_2(M^N f_1,\ M^N f_2) \le hD_2(f_1, f_2)$, where $f_1, f_2 \in S$, $0 < h < 1$. From that S is a complete space, we know that the mapping M is a compressed mapping.
So M has and only has a fixed point in S such that

$$Mf(x) = f(x),\ x \in [x_0, x_1]$$
$$f(x) = g(x) + \lambda \int_{x_0}^{x} K(x,t) f(t) dt$$

Therefore, $f(x)$ is the unique continuous solution of integral equation (I). In the same time, we know that $f(x)$ can be obtained with successive iteration method. #

3 The Initial Value Problem of FDE

In this section, we study the initial value of FDE:

$$\frac{df(x)}{dx} = K(x, f(x)) \qquad f(x_0) = m \qquad \text{...............} (3.1)$$

where $f(x) \in F$ is a unknown fuzzy function. The mapping $K : [x_0, x_1] * F \to F$ is continuous, and $0 < x_1 - x_0 < 1$. m is a fuzzy number.

Theorem 3.1 Suppose that

1. fuzzy integral $\int_{x_0}^{x} K(t, f(t))dt$, $x \in [x_0, x_1]$ exists for any fuzzy function $f(x)$,
2. for any $f_1(x) = (a_1(r,x), b_1(r,x)) \in S, f_2(x) = (a_2(r,x), b_2(r,x)) \in S,$
   $K(x, f_i) = (Ka_i(r,t),\ Kb_i(r,t)) \qquad i = 1, 2.$
   $|Ka_1(r,x) - Ka_2(r,x)| \le |a_1(r,x) - a_2(r,x)|$
   $|Kb_1(r,x) - Kb_2(r,x)| \le |b_1(r,x) - b_2(r,x)|$

Then the initial problem of FDE (3.1) has unique continuous solution on $[x_0, x_1]$.

Proof. The initial problem (3.1) is equivalent to the integral equation

$$f(x) = m + \int_{x_0}^{x} K(t, f(t))dt \qquad \text{...............} (3.2)$$

Let mapping $M : S_{[x_0,x_1]} \to S_{[x_0,x_1]}$ such that for any $f(x) \in S_{[x_0,x_1]}$,

$$Mf(x) = m + \int_{x_0}^{x} K(t, f(t))dt$$

For any $f_1, f_2 \in S_{[x_0,x_1]}$ and any $x \in [x_0, x_1]$

$$\begin{aligned}
&D(Mf_1(x),\ Mf_2(x)) \\
&= sup_{r\in I}\ max\{|\int_{x_0}^{x}(Ka_1(r,t) - Ka_2(r,t))dt|,\ |\int_{x_0}^{x}(Kb_1(r,t) - Kb_2(r,t))dt|\} \\
&\le sup_{r\in I}\ max\{\int_{x_0}^{x} |Ka_1(r,t) - Ka_2(r,t)|dt,\ \int_{x_0}^{x} |Kb_1(r,t) - Kb_2(r,t)|dt\} \\
&\le sup_{r\in I}\ max\{\int_{x_0}^{x} |a_1(r,t) - a_2(r,t)|dt,\ \int_{x_0}^{x} |b_1(r,t) - b_2(r,t)|dt\} \\
&\le (x - x_0) sup_{r\in I}\ max\{max_{t\in[x,x_0]}|a_1(r,t) - a_2(r,t)|,\ max_{t\in[x,x_0]}|b_1(r,t) - b_2(r,t)|\} \\
&= (x - x_0) max_{t\in[x,x_0]} sup_{r\in I}\ max\{|a_1(r,t) - a_2(r,t)|,\ |b_1(r,t) - b_2(r,t)|\} \\
&\le (x - x_0) max_{t\in[x_0,x]} D(f_1(t),\ f_2(t))
\end{aligned}$$

So,

$$\begin{aligned}
&D_2(Mf_1,\ Mf_2) \\
&= sup_{x\in[x_0,x_1]} D(Mf_1(x),\ Mf_2(x)) \le sup_{x\in[x_0,x_1]}\{(x - x_0) max_{t\in[x_0,x]} D(f_1(t),\ f_2(t))\} \\
&= (x_1 - x_0) D_2(f_1, f_2)
\end{aligned}$$

Since $0 < x_1 - x_0 < 1$ and using Banach compressed mapping principle, we know that the mapping M has and only has a fixed point $f(x)$:

$$f(x) = m + \int_{x_0}^{x} K(t, f(t))dt.$$

$f(x)$ is the unique continuous solution of the initial problem (3.1). #

REFERENCES

[1] Wang, Y. and Wu, S., Fuzzy Differential Equations, *Proceedings of Second International Fuzzy System Association Congress*, Tokyo, Japan, Vol.1, 298-301, 1987.

[2] Wang, Y., Fuzzy Representation of Eiotic Population Growth, *Proceedings of Third International Fuzzy System Association Congress*, Seattle, WA., 524-527, 1989.

[3] Goetschel, R. and Voxman, W., Topological Properties of Fuzzy Numbers, *Fuzzy Sets and Systems*, 18(1986), 31-43, 1986.
[4] Rhoades, B. H., A Comparison of Various Definitions of Contractile Mapping, *Transactions of the American Mathematics Society*, Vol.226, 1977.