FUZZY OBSERVABLES AND FUZZY RANDOM VARIABLES

Radko MESIAR, Slovac Technical University, Radlinského 11, 813 68 Bratislava, Czechoslovakia

Krzysztof PIASECKI, Dept. of Mathematics, Academy of Economics, ul. Marchlewskiego 146/150, 60-967 Poznań, Poland

1. Introduction

Let $(\Omega, \mathcal{S})$ be a measurable space, where $\mathcal{S}$ is a σ-algebra of crisp subsets of Ω. A random variable f is a mapping, $f: \Omega \longrightarrow R$, whose inverse f^{-1} is a σ-homomorphism of Borel subsets of real line to $\mathcal{S}$, $f^{-1}: \mathcal{B} \longrightarrow \mathcal{S}$,

$$f^{-1}/E^c/ = (f^{-1}/E/)^c \quad \text{and}$$

$$f^{-1}/\bigcup E_i/ = \bigcup f^{-1}/E_i/ \quad .$$

There are several ways how to generalize the notion of a random variable to the fuzzy case. We'll deal with two accesses. In the first one we generalize the underlaying measurable space and we deal with the inverse f^{-1} generalization. This access was used by Dvurečenskij and Riečan in [1] and by Mesiar in [4]. The main idea of the second access is a generalization of the real numbers. This approach was used by Klement in [3].

The main purpose of this paper is to show that the first approach is / via representation / a special case of the second one.

2. Fuzzy observables

Let (Ω, M) be a fuzzy quantum space, see [1], i.e. M is a soft fuzzy σ-algebra of fuzzy subsets of Ω,

$$0_\Omega \in M \qquad /1/$$

$$\mu \in M \Longrightarrow \mu' = 1 - \mu \in M \qquad /2/$$

$$\{\mu_i\} \subset M \Longrightarrow \vee \mu_i = \sup\{\mu_i\} \in M \qquad /3/$$

$$/\tfrac{1}{2}/_\Omega \notin M \qquad /4/$$

Dvurečenskij and Riečan in [1] have defined a fuzzy observable x as a σ-homomorphism, $x: \mathcal{B} \rightarrow M$,

$$x/E^c/ = (x/E/)' \qquad /5/$$

$$x/\cup E_i/ = \vee\, x/E_i/ \qquad /6/$$

We recall here some results - for more details see [1,2] .

<u>Proposition 2.1.</u> x is a fuzzy observable on (Ω, M) iff the system $\{b/t/, t \in R\}$, $b/t/ = x/]-\infty, t[/$, has the following properties:

$$t \le s \Longrightarrow b/t/ \le b/s/ \qquad /7/$$

$$\Big(\bigvee_{t \in R} b/t/\Big)' = \bigwedge_{t \in R} b/t/ \qquad /8/$$

$$\bigvee_{t < s} b/t/ = b/s/ \quad \forall s \in R \qquad /9/$$

$$b/t/ \vee (b/t/)' = b/s/ \vee (b/s/)' \quad \forall t, s \in R \qquad /10/$$

Conversely, if a system $\{b/t/, t \in R\} \subset M$ has properties /7/-/10/, then there exists unique fuzzy observable x such that $x/]-\infty, t[/ = b/t/$ for any real t ∎

<u>Definition 2.1.</u> Let x, y be two fuzzy observables on (Ω, M). Then their sum $z = x + y$ is uniquelly determined by the system $\{z/]-\infty, t[/, t \in R$, where

$$z/]-\infty, t[/ = \bigvee_{r \in Q} (x/]-\infty, r[/ \wedge y/]-\infty, t-r[/), \qquad /11/$$

Q is the set of all rational numbers ∎

3. Fuzzy-valued random variables

Klement in [3] presents a concept of the fuzzy real line.

A nonfuzzy real number r is there identified with the Dirac distribution $\delta_r = 1_{]r,+\infty[}$. A fuzzy number p is a function, $p: \overline{R} \longrightarrow [0, 1]$, $\overline{R} = R \cup \{-\infty, +\infty\}$, such that

$$p/-\infty/ = 0 \quad \text{and} \quad p/+\infty/ = 1 \qquad /12/$$

$$\forall r \in R: \quad p/r/ = \sup\{p/s/, \ s < r\} \quad . \qquad /13/$$

A natural interpretation of a fuzzy number p is the following: p/r/ is the degree to which p is less than the /nonfuzzy/ number r . For the fuzzy arithmetics are used the quasi-inverses p^i of fuzzy numbers p, $p^i: [0, 1] \longrightarrow \overline{R}$,

$$p^i/0/ = -\infty \qquad /14/$$

$$p^i/s/ = \sup\{r \in \overline{R}, \ p/r/ < s\} \ , \quad s \in]0, 1] \qquad /15/$$

Then

$$p \le q \Longleftrightarrow p^i/a/ \le q^i/a/ \qquad \forall a \in [0, 1] \qquad /16/$$

$$p/r/ = \sup\{a \in [0, 1], \ p^i/a/ < r\} \quad \forall r \in R \qquad /17/$$

$$(p \oplus q)^i/a/ = p^i/a/ + q^i/a/ \quad \forall a \in [0, 1] \qquad /18/$$

The /extended/ fuzzy real line $\overline{R}$/I/ is the set of all fuzzy numbers p. $\overline{R}$/I/ can be embedded naturaly into $[0, 1]^{\overline{R}}$ equipped with the product σ-algebra.

<u>Definition 3.1.</u> Let $(\Omega, \mathcal{S})$ be a measurable space. Any measurable function X , $X: \Omega \longrightarrow \overline{R}$/I/ will be called fuzzy-valued random variable, briefly fuzzy random variable.

For more details see [3] ■

4. Fuzzy random variable representation of fuzzy observables

Let a fuzzy quantum space (Ω, M) be given. Then it induces a classical measurable space $(\Omega, K/M/)$, see [5] , where

$$K/M/ = \{A \subset \Omega, \exists \mu \in M, \{\mu > \tfrac{1}{2}\} \subseteq A \subseteq \{\mu \ge \tfrac{1}{2}\}\}. \qquad /19/$$

Let x be a fuzzy observable. Let $\omega \in \Omega$ and $r \in R$. Then we can take

$$x/]-\infty, r[\,//\omega/ = x/\omega, r/ \qquad /20/$$

as the degree to which the "number" $x/\omega/$ is less than r. For $\pm\infty$ we define $x/\omega, -\infty/ = 0$ and $x/\omega, +\infty/ = 1$. Similar approach in the case of classical random variable leads to the nonfuzzy real number $f/\omega/$, namely to its representation in $\bar{R}/I/$ by $\delta_{f/\omega/}$.

<u>Proposition 4.1.</u> For $\forall\omega\in\Omega$, $x/\omega/$ is a fuzzy number of $\bar{R}/I/$.

Proof: It is enough to show for every real r the validity of /13/, i.e. $x/\omega, r/ = \sup\{x/\omega, s/, s < r\}$. But this is implied by /9/ in Proposition 2.1.

<u>Proposition 4.2.</u> Let x be a fuzzy observable. Then there exist a random variable f on the space $(\Omega, K/M/)$ and a fuzzy subset $v \in M$, $v \geq \frac{1}{2}$, such that, for every $\omega \in \Omega$, the fuzzy number $x/\omega/$ is of the next form:

$$x/\omega, r/ = \begin{matrix} 1 - v/\omega/ & \text{if } r \leq f/\omega/, r \in R \\ v/\omega/ & \text{if } r > f/\omega/, r \in R \end{matrix} \quad . /21/$$

Proof: We use /7/-/10/ of Proposition 2.1. /10/ implies $x/\omega, t/ \vee (x/\omega, t/)' = x/\omega, s/ \vee (x/\omega, s/)' = x/R//\omega/$ for any $t, s \in R$. Denote $v = x/R/ \in M$. It is easy to see that $v \geq \frac{1}{2}$. Then $x/\omega, r/ = v/\omega/$ or $x/\omega, r/ = 1 - v/\omega/$ for every real r. If $x/\omega, r/ = v/\omega/$ for all $r \in R$, then /8/ implies $v/\omega/ = v'/\omega/$, so that $v/\omega/ = \frac{1}{2}$. Similar is the case $x/\omega, r/ = 1 - v/\omega/$ for all $r \in R$. If $v/\omega/ > \frac{1}{2}$, then /7/ and /9/ imply the existence of a real number $f/\omega/$ such that /2/ holds. For those $\omega \in \Omega$, for which $v/\omega/ = \frac{1}{2}$, we define $f/\omega/ = 0$.

The measurability of f follows from the next facts:

$$x/f/\omega///\omega/ = v/\omega/ \qquad /22/$$

$$x/E/\in M \quad \forall E \in \mathcal{B} \qquad /23/$$

$$f^{-1}/E/ = \{\omega\in\Omega,\ f/\omega/\in E\} = \begin{cases} \{\omega,\ x/E//\omega/ > \frac{1}{2}\} & \text{if } 0\notin E \\ \{\omega,\ x/E//\omega/ \geq \frac{1}{2}\} & \text{if } 0\in E \end{cases} \qquad /24/$$

Now, it is easy to see that $f^{-1}/E/\in K/M/$ for any Borel subset E .

<u>Corollary 4.1.</u> Let x be a fuzzy observable on a fuzzy quantum space (Ω , M) . Then the function $X:\Omega \longrightarrow \overline{R}/I/$, $X/\omega/ = x/\omega/ \ \forall\omega\in\Omega$, is a fuzzy random variable on $(\Omega,\ 2^{\Omega})$.

<u>Corollary 4.2.</u> Let (Ω, M) be a fuzzy quantum space and X let be a fuzzy random variable on $(\Omega,\ 2^{\Omega})$ such that there exist a random variable f on $(\Omega , K/M/)$ and a fuzzy subset $v\in M$, $v \geq \frac{1}{2}$, so that for each $\omega\in\Omega$, $r\in R$, is satisfied /21/ for $X/\omega//r/$. Then there exists unique fuzzy observable x on (Ω , M) such that $x/\omega/ = X/\omega/$.

Corollaries 4.1. and 4.2. show that the approach of Dvurečenskij and Riečan is a special case of the Klement's one. Similar is the situation if we take into account a general fuzzy observable defined by Mesiar in [4] . Presented representation preserves the algebraic structure .

<u>Proposition 4.3.</u> Let x, y be two fuzzy observables on a fuzzy quantum space (Ω , M) and let $z = x + y$. Then for corresponding fuzzy random variables we have $Z = X \oplus Y$.

Proof: Due to Proposition 2.1. it is enough to prove

$Z/\omega//t/ = z/\omega , t /$ for any $t\in R$. We have

$$Z/\omega//t/ = \sup\{ a\in[0, 1] ,\ Z^{i}/\omega//a/ < t\} =$$

$$= \sup\{ a\in[0,1] ,\ X^{i}/\omega//a/ + Y^{i}/\omega//a/ < t\} =$$

$$= \sup \{ a \in [0, 1] , \sup \{ r, X/\omega//r/ < a \} + \sup \{ s, Y/\omega//s/ < a \} < t \} . \qquad /25/$$

a in the brackets cannot be greater then $v_x/\omega/$, as in that case $\sup \{ r, X/\omega//r/ < a \} = +\infty$. Similarly $a \leq v_y/\omega/$, so that $a \leq (v_x \wedge v_y)/\omega/$. If $a = (v_x \wedge v_y)/\omega/$, then $\sup \{ r, X/\omega//r/ < a \} + \sup \{ s, Y/\omega//s/ < a \} = f_x/\omega/ + f_y/\omega/$.

It follows

$$Z/\omega//t/ = (v_x \wedge v_y)/\omega/ \quad \text{for } t > f_x/\omega/ + f_y/\omega/ .$$

Analogously for $t \leq f_x/\omega/ + f_y/\omega/$ we get

$$Z/\omega//t/ = 1 - (v_x \wedge v_y)/\omega/ .$$

On the other hand, by Definition 2.1. we have defined the sum z. We have

$$z/\omega, t/ = \bigvee_{r \in Q} (X/\omega//r/ \wedge Y/\omega//t-r/) = \begin{cases} 1 - (v_x \wedge v_y)/\omega/ & \text{if } t \leq f_x/\omega/ + f_y/\omega/ \\ v_x \wedge v_y /\omega/ & \text{if } t > f_x/\omega/ + f_y/\omega/ \end{cases} .$$

All these facts imply the Proposition.

<u>Remark 4.1.</u> In the similar way we can show the existence of the product of two fuzzy observables, $z = x.y$. In this case we have $Z = X \odot Y$, $f_z = f_x . f_y$ and $v_z = v_x \wedge v_y$.

<u>References</u>

[1] Dvurečenskij, A., Riečan, B.: On joint observables for F-quantum space. Busefal 35, 1988, 10-14.

[2] Dvurečenskij, A., Tirpáková, A.: A note on a sum of observables in F-quantum spaces and its properties. Busefal 35, 1988, 132-137.

[3] Klement, E.P.: Strong law of large numbers for random variables with values in the fuzzy real line. Communications IFSA, 1987, Math. Chapt., 7-11.

[4] Mesiar, R.: A note to the fuzzy observables. Busefal 40, 1989, 50-53 .

[5] Piasecki, K.: Some new results for fuzzy P-measures. Busefal 33, 1987, 76-85.