# THE AUTOCONTINUITY OF THE FUZZY NUMBER-VALUED FUZZY MEASURE*

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Let X be a non-empty set, F(X) be a $\sigma$-algebra of subsets of X, R be a set of all real-numbers. F(R) be a set of all fuzzy numbers[1].

By decompsition theorem of fuzzy set, for every $\tilde{a} \in F(R)$,

$$a = \bigcup_{\lambda\in[0,1]} \lambda \cdot a_\lambda ,$$

where $a_\lambda = \{x;\ \tilde{a}(x) \geq \lambda\}$ is a closed interval, denoted by $[a_\lambda^-,\ a_\lambda^+]$.

Definition 1. For $\tilde{a}, \tilde{b} \in F(R)$, we say that $\tilde{a} \leq \tilde{b}$, if for every $\lambda \in (0\ ,\ 1]$, $a_\lambda^- \leq b_\lambda^-$ and $a_\lambda^+ \leq b_\lambda^+$. We say that $\tilde{a} < \tilde{b}$, if $\tilde{a} \leq \tilde{b}$ and there exists $\lambda_0 \in (0\ ,\ 1]$ such that $a_{\lambda_0}^- < b_{\lambda_0}^-$ or $a_{\lambda_0}^+ < b_{\lambda_0}^+$. We say that $\tilde{a} = \tilde{b}$, if $\tilde{a} \leq \tilde{b}$ and $\tilde{b} \leq \tilde{a}$.

Definition 2. For every positive real-number M, there exists $\lambda \in (0\ ,\ 1]$ such that $M < a_\lambda^+$ or $a_\lambda^- < -M$, then $\tilde{a}$ is called fuzzy infinity, written $\tilde{\infty}$.

Let $F_+(R) = \{\tilde{a};\ \tilde{a} \geq 0,\ \tilde{a} \in F(R)\}$.

Definition 3. $\tilde{\rho}(\tilde{a}\ ,\ \tilde{b})$ defined by the equality (*) is called a fuzzy distance of fuzzy numbers $\tilde{a}$ and $\tilde{b}$,

$$\tilde{\rho}(\tilde{a}\ ,\ \tilde{b}) = \bigcup_{\lambda\in[0,1]} \lambda[\,|a_1^- - b_1^-|\ ,\ \sup_{\lambda\leq\eta\leq1} |a_\eta^- - b_\eta^-| \vee |a_\eta^+ - b_\eta^+|\,], \qquad (*)$$

It is easy to see that, if $a, b \in R$, then $\tilde{\rho}(a\ ,\ b) = |a - b|$.

Definition 4. Let $\{\tilde{a}_n\} \subset F(R)$, $\tilde{a} \in F(R)$, $\{\tilde{a}_n\}$ is said to converges to $\tilde{a}$ in fuzzy distance $\tilde{\rho}$, denoted by $\lim_{n\to\infty} \tilde{a}_n = \tilde{a}$ $(\tilde{a}_n \longrightarrow \tilde{a})$, if for arbitrary $\varepsilon > 0$, there exists an integer $N > 0$ such that

$$\tilde{\rho}(\tilde{a}_n , \tilde{a}) < \varepsilon \qquad \text{as } n \geq N.$$

Theorem 1. Let $\{\tilde{a}_n\} \subset F(R)$, $\tilde{a} \in F(R)$, then $\{\tilde{a}_n\}$ converges to $\tilde{a}$ in fuzzy distance $\tilde{\rho}$ if and only if $\{a^-_{n_\lambda}\}$, $\{a^+_{n_\lambda}\}$ converges to $a^-_\lambda$, $a^+_\lambda$ uniformly for every $\lambda \in (0 , 1]$ in usual distance of real-numbers.

Definition 5. Fuzzy number-valued fuzzy measure ((z)fuzzy measure) on $F(X)$ is a fuzzy number-valued set function $\mu$: $F(X) \longrightarrow F_+(R)$, with the properties:

ZFM1. $\mu(\emptyset) = 0$;

ZFM2. $A \subset B$, $\Longrightarrow \mu(A) \leq \mu(B)$;

ZFM3. $A_n \nearrow$ , $\{A_n\} \subset F(X)$, $\Longrightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$;

ZFM4. $A_n \searrow$ , $\{A_n\} \subset F(X)$ and there exists $n_0$ such that $\mu(A_{n_0}) \neq \tilde{\infty}$, $\Longrightarrow \mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

$(X , F(X))$ is called a measurable space, $(X , F(X), \mu)$ is called a (z)fuzzy measure space.

Definition 6. (z)fuzzy measure $\mu$ is called autocontinuous from above(resp. from below), if we have $\lim_{n\to\infty} \mu(A \cup B_n) = \mu(A)$ (resp. $\lim_{n\to\infty} \mu(A - B_n) = \mu(A)$, whenever $A \in F(X)$, $\{B_n\} \subset F(X)$, and $\lim_{n\to\infty} \mu(B_n) = 0$, u is called autocontinuous, if it is both autocontinuous from above and autocontinuous from below.

Definition 7. 1) (z)fuzzy measure $\mu$ is called null-additive, if we have $\mu(A \cup B) = \mu(A)$, whenever $A \in F(X)$, $B \in F(X)$ with $\mu(B)=0$;

2) (z)fuzzy measure $\mu$ is called subadditive, if we have $\mu(A \cup B) \leq \mu(A) + \mu(B)$, whenever $A, B \in F(X)$.

Theorem 2. 1) (z)fuzzy measure $\mu$ is subadditive, then it is autocontinuous; 2) If (z)fuzzy measure $\mu$ is autocontinuous from above, then it is null-additive.

Theorem 3. If (z)fuzzy measure $\mu$ is autocontinuous, then we have $\lim_{n\to\infty}\mu(A \triangle B_n) = \mu(A)$, for any $A \in F(X)$, $\{B_n\} \subset F(X)$ with $\lim_{n\to\infty}\mu(B_n) = 0$.

Theorem 4. Let $\{E_n\} \subset F(X)$, if (z)fuzzy measure $\mu$ is autocontinuous from above, and $\lim_{n\to\infty}\mu(E_n) = 0$, then there exists some sequence $\{E_{n_i}\}$ of subsequences of $\{E_n\}$, such that $\mu(\bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty} E_{n_i}) = 0$

Definition 8. A mapping $f: X \longrightarrow R$ is called fuzzy measurable function, if $F_\alpha = \{x;\ f(x) \geq \alpha,\ \alpha \in R\}$. The set of all fuzzy measurable functions is denoted by $\underline{M}$.

Definition 9. Let $A \in F(X)$, $\{f_n\} \subset \underline{M}$, $f \in \underline{M}$, 1) If there exists $D \subset X$ with $\mu(D) = 0$ such $\{f_n\}$ converges to f on A-D, then we say "$\{f_n\}$ converges to f almost everywhere on A"; 2) If $\mu(A \cap \{|f_n - f| \geq \varepsilon\}) \longrightarrow 0$, for any given $\varepsilon > 0$, then we say $\{f_n\}$ converges in (z)fuzzy measure $\mu$ to f on A. We denote $f_n \xrightarrow[A]{a.e.} f$, $f_n \xrightarrow[A]{\mu} f$ respectively.

Theorem 5. (Riesz's theorem) Let $\{f_n\} \subset \underline{M}$, $f \in \underline{M}$, $A \in F(X)$, if $\mu$ is autocontinuous from above, $f_n \xrightarrow[X]{u} f$, then there exists some subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that $f_{n_i} \xrightarrow[A]{a.e.} f$.

Theorem 6. (Lebesgue's theorem) Let $\{f_n\} \subset \underline{M}$, $f \in \underline{M}$, $A \in F(X)$, if $f_n \xrightarrow[A]{a.e.} f$, and $\mu(A) \neq \tilde{\infty}$, then $f_n \xrightarrow[A]{\mu} f$.

Theorem 7. (Egoroff's theorem) If $\mu$ is autocontinuous from above and $\{f_n\} \subset \underline{M}$, $f \in \underline{M}$, $A \in F(X)$, $\mu(A) \neq \tilde{\infty}$, and $f_n \xrightarrow[A]{a.e.} f$, then for any $\varepsilon > 0$, there exists $E \in F(X)$ with $\mu(E) < \varepsilon$, such that $\{f_n\}$ converges to f uniformly on A - E, where f is an a.e. finite-valued measurable function on A.

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