ON ONE PROBLEM OF SET-VALUED MEASURES

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1. Introduction

This paper is a contribution to the theory of set-valued measures and q-σ-algebras. Our main goal is to prove a version of Costé's theorem ([2]) about the convexity of the closure of values of nonatomic set-valued measure of bounded variation on q-σ-algebra.

Throughout this paper, let X be a nonvoid abstract set. The symbol $\mathcal{A}$ will stand for a q-σ-algebra of subsets of X, i.e. $\mathcal{A}$ is the class of subsets of X with properties:

i) $X \in \mathcal{A}$,

ii) $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$,

iii) $A_i \in \mathcal{A}$, $A_i \cap A_j = \emptyset$, $i \neq j = 1,2,\ldots$, implies $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

In this context A^c means the complement of the set $A \subset X$. Let Z be a nonempty system of subsets of X. Let $s(Z)$, $\sigma(Z)$ and $\sigma_q(Z)$ denote the algebra, σ-algebra and q-σ-algebra generated by Z, respectively.

The symbol Y will stand for a real Banach space with norm $\|\cdot\|$ and let 2^Y be the family of all nonempty subsets of Y. Denote by clB the closure of $B \in 2^Y$.

A Banach space Y is said to have the Radon-Nikodym property (RNP) if for each finite measure space $(\Omega, \mathcal{S}, \nu)$ and each ν-continuous Y-valued measure $m: \mathcal{S} \to Y$ of bounded variation, there exists a Bochner integrable function $f: \Omega \to Y$ such that $m(A) = \int_A f\, d\nu$ for all $A \in \mathcal{S}$. Dunford and Pattis [4] and Phillips [7] showed, that every separable dual space and every reflexive space has the RNP.

For $B \subset Y$ let us define the number $\|B\|$ by

$$\|B\| = \sup_{b \in B} \|b\| .$$

A set-valued function $M: \mathcal{A} \to 2^Y$ is said to be countably additive if $M(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} M(A_i)$ for every sequence $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint elements of $\mathcal{A}$, where given a sequence $\{B_i\}_{i=1}^{\infty}$ in 2^Y the sum $\sum_{i=1}^{\infty} B_i$ is defined as follows:

$$\sum_{i=1}^{\infty} B_i = \{ y \in Y:\ y = \sum_{i=1}^{\infty} y_i \ \text{(unconditionaly convergent)},\ y_i \in B_i,\ i \geq 1 \} .$$

A map $M: \mathcal{A} \to 2^Y$ is said to be a set-valued measure if M is countably additive and $M(\emptyset) = \{0\}$.

Let $M: \mathcal{A} \to 2^Y$ be a set- valued measure. For each $A \in \mathcal{A}$ define

$$\|M\|(A) = \sup \sum_{i=1}^{n} \|M(A_i)\|$$

where the supremum is taken over all finite partitions $\{A_1, A_2, \ldots, A_n\}$ of A. A set-valued measure M is said to be of bounded variation if $\|M\|(X) < \infty$.

An element $A \in \mathcal{A}$ is said to be an atom of a set-valued measure $M: \mathcal{A} - 2^Y$, if $M(A) \neq \{0\}$ and if either $M(B) = \{0\}$ or $M(A \setminus B) = \{0\}$ holds for every $B \subset A$, $B \in \mathcal{A}$. A set-valued measure having no atoms is said to be nonatomic.

In our further considerations takes an important place the notion dyadic structure of a set $A \in \mathcal{A}$. The notation of the dyadic structure of $A \in \mathcal{A}$ is a collection of sets $A(\varepsilon_1 \varepsilon_2 \ldots \varepsilon_k) \in \mathcal{A}$, where $\varepsilon_i = 0,1$ and $k=1,2,\ldots$ such that

$$A(0) \cup A(1) = A \qquad \text{and} \qquad A(\varepsilon_1 \ldots \varepsilon_k 0) \cup A(\varepsilon_1 \ldots \varepsilon_k 1) = A(\varepsilon_1 \ldots \varepsilon_k),$$

$$A(0) \cap A(1) = \emptyset \qquad \text{and} \qquad A(\varepsilon_1 \ldots \varepsilon_k 0) \cap A(\varepsilon_1 \ldots \varepsilon_k 1) = \emptyset.$$

Theorem 1. Assume that Y has the RNP. Let $M: \mathcal{A} \to 2^Y$ be a non-atomic set-valued measure of bounded variation. Then $\mathrm{cl}M(A)$ is convex for every $A \in \mathcal{A}$.

Before starting the proof of Theorem 1 we shall formulate a few propositions and lemmas.

Lemma 2. Suppose that for X there exists a dyadic structure $D_0 = \{ A(\varepsilon_1 \ldots \varepsilon_k),\ k \geq 1 \}$. Denote $D = D_0 \cup \{\emptyset\}$ and $\mathcal{A}_0 = s(D)$. Then

$$\mathcal{A}_0 = \left\{ \bigcup_{i=1}^{n} A_i : A_i \in D,\ n \geq 1 \right\} \text{ holds.}$$

Proof. Let $P = \left\{ \bigcup_{i=1}^{n} A_i : A_i \in D,\ n \geq 1 \right\}$. It is obvious that $\mathcal{A}_0 \supset P$.

Conversely it is sufficient to show that

i) $A,B \in P$ implies $A \cup B \in P$,

ii) $A \in P$ implies $A^c \in P$.

The proof of i) is trivial. From the construction of the dyadic structure D follows, that $A \cap B \in P$ for $A,B \in P$. Let $A \in P$ be arbitrary, then $A = \bigcup_{i=1}^{n} A_i$, $A_i \in D$ where $A_i = A(\varepsilon_1^i \ldots \varepsilon_{k_i}^i)$

for certain $k_i \geq 1$ $(i=1,2,\ldots,n)$. Then it is easy to show that

$A_i^c = \bigcup_{j=1}^{k_i} A(\varepsilon_1^i \ldots \varepsilon_{j-1}^i \delta_j^i)$, where $\delta_j^i = 1 - \varepsilon_j^i$ $(j=1,2,\ldots,k_i)$. From this we can derive, that $\mathcal{A}_0 \subset P$. This concludes the proof.

We know ([6]), that $\sigma_q(Z) = \sigma(Z)$ iff $E \cap F \in \sigma_q(Z)$ for all $E,F \in Z$.

From Lemma 2 the following proposition follows.

Proposition 3. Let $\mathcal{A}_1 = \sigma(\mathcal{A}_0)$. Then $\mathcal{A}_1 \subset \mathcal{A}$.

Proposition 4. Let $M: \mathcal{A} \to 2^Y$ be a set-valued measure. Then the set function $\|M\|$ defined by (1) is non-decreasing on $\mathcal{A}$.

Proof. Let $A, B \in \mathcal{A}$, $A \supset B$ and $\{B_1, \dots, B_n\} \subset \mathcal{A}$ be an arbitrary partition of B. Putting $B_0 = A \setminus B$ we obtain $B_0 \in \mathcal{A}$, $B_0 \cap B_i = \emptyset$ for all $i = 1, 2, \dots, m$, $A = \bigcup_{i=0}^{n} B_i$, thus $\{B_i\}_{i=0}^{n}$ is a partition of A. It means, that the followint holds:

$$\sum_{i=1}^{n} \| M(B_i) \| < \sum_{j=0}^{n} \| M(B_j) \| \leq \|M\| (A), \quad \text{i.e.} \quad \|M\|(B) \leq \|M\|(A).$$

Lemma 5. Let ν be a measure on a ring $\mathcal{R}$ and ν^* the Caratheodory extension of ν to $\sigma(\mathcal{R})$. Then ν^* has no atom of finite measure iff the following condition holds:

For each $A \in \mathcal{R}$ with $0 < \nu(A) < \infty$ the set $\{\nu(E) : E \in \mathcal{R},\ E \subset A\}$ is dense in the interval $\langle 0, \nu(A) \rangle$.

Lemma 5 is proved in [1].

Proof of Theorem 1.

Without loss of generality we may assume $A = X$ and that $M(X) \neq \{0\}$. It follows from the nonatomicity of M, that there exists a set $A(0) \in \mathcal{A}$ such, that $\emptyset \subsetneqq A(0) \subsetneqq X$ and $M(A(0)) \neq \{0\}$, $M(X \setminus A(0)) \neq \{0\}$. Denote by $A(1) = X \setminus A(0)$. Then holds $A(0) \cup A(1) = X$, $A(0) \cap A(1) \neq 0$ and $A(1) \in \mathcal{A}$. The repetition of the procedure just described will give us a dyadic structure $D_0 = \{A(\varepsilon_1 \varepsilon_2 \dots \varepsilon_k),\ k \geq 1\}$ for X such, that for every set $A \in D_0$ holds $A \in \mathcal{A}$ and $M(A) \neq \{0\}$.

To show the convexity of $\mathrm{cl}M(X)$ it is sufficient to prove that if $x_1, x_2 \in M(X)$ and $0 < \alpha < 1$, then $\alpha x_1 + (1-\alpha) x_2 \in \mathrm{cl}M(X)$.

So choose arbitrary $x_1, x_2 \in M(X)$ and $0 < \alpha < 1$. We can find elements $x_1(\varepsilon_1 \ldots \varepsilon_k)$, $x_2(\varepsilon_1 \ldots \varepsilon_k) \in Y$ where $\varepsilon_i \in \{0,1\}$ and $k=1,2,\ldots$, such that

$$\begin{aligned}
&\text{i)}\ x_j(\varepsilon_1 \ldots \varepsilon_k) \in \mathcal{M}(A(\varepsilon_1 \ldots \varepsilon_k)), \\
&\text{ii)}\ x_j(0) + x_j(1) = x_j, \qquad\qquad (2)\\
&\text{iii)}\ x_j(\varepsilon_1 \ldots \varepsilon_k 0) + x_j(\varepsilon_1 \ldots \varepsilon_k 1) = x_j(\varepsilon_1 \ldots \varepsilon_k) \qquad (j=1,2).
\end{aligned}$$

In Proposition 3 we have shown, that putting $\mathcal{A}_0 = s(D)$, $\mathcal{A}_1 = \sigma(\mathcal{A}_0)$ holds that

$$\mathcal{A}_1 \subset \mathcal{A}. \qquad (3).$$

This is a very useful relation, because the most of further considerations will be based on the fact, that in consequence of (3) we can work on the σ-algebra $\mathcal{A}_1$.

Taking in view the structure of $\mathcal{A}_0$ and relations i)-iii) in (2) we can define additive set-functions $m_j : \mathcal{A}_0 \to Y$ $(j=1,2)$ as $m_j(A) = x_j(A)$ for $A \in D_0$ and $m_j(A) = \sum_{i=1}^{n} m_j(A_i)$ for $A = \bigcup_{i=1}^{n} A_i \in \mathcal{A}_0$, $A_i \in D$.

Let us define a set-function $\mu : \mathcal{A}_1 \to \langle 0, \infty\rangle$ as

$\mu(A) = \sup\limits_{\pi A} \sum_{i=1}^{n} \| M(A_i) \|$ for $A \in \mathcal{A}_1$, where the supremum is taken over the partitions $\pi_A = \{A_1, \ldots, A_n : A_i \in \mathcal{A}_1\ (i=1,\ldots,n),\ A_i \cap A_j = \emptyset$ for $i \neq j$, $A = \bigcup_{i=1}^{n} A_i$, $n \geq 1\}$. In [5] there is shown that μ is a measure on $\mathcal{A}_1$.

It follows that $\mu(A) \leq \|M\|(A)$ for every $A \in \mathcal{A}_1$. Since M is of bounded variation and $\|M\|$ is non-decreasing, we conclude, that μ is a finite measure on $\mathcal{A}_1$.

We know, that $\mathcal{A}_0$ is dense in $\mathcal{A}_1$ in the topology induced by the pseudo-metric $\rho : \mathcal{A}_1 \times \mathcal{A}_1 \to \langle 0, \infty \rangle$ defined by

$\rho(A,B) = \mu(A \triangle B)$ for $A, B \in \mathcal{A}_1$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ ([6]).

There holds, that

$\| m_j(A)\| \le \mu(A)$ for $A \in \mathcal{A}_0$ $(j=1,2)$. In this case ([3]) there exist extensions $\tilde{m}_j : \mathcal{A}_1 \longrightarrow Y$ of the set functions $m_j : \mathcal{A}_0 \to Y$ such, that $\tilde{m}_j$ are vector measures on $\mathcal{A}_1$ and

$\| \tilde{m}_j(A) \| \le \mu(A)$ for every $A \in \mathcal{A}_1$ (4).

Denote these extensions by the same symbols m_j.

Let us show, that μ is nonatomic. By Lemma 5 we need to show the following condition:

if $A \in \mathcal{A}_0$, $0 < \mu(A) < \infty$, then for every $\varepsilon > 0$ there exists $B \in \mathcal{A}_0$, $B \subset A$ such, that

$0 < \mu(A) - \mu(B) < \varepsilon$

So let $A \in \mathcal{A}_0$ be arbitrary, $0 < \mu(A) < \infty$. From Lemma 2 it is clear, that there exist $n \geqslant 1$, $\{A_i\}_{i=1}^n$, $A_i \in D$ such that $A = \bigcup_{i=1}^n A_i$, $A_i \cap A_j = \emptyset$ for $i \neq j = 1,2,\ldots,n$. Assume, that $A_1 \neq \emptyset$ and for every $\varepsilon > 0$ find a set $B_1 \in \mathcal{A}_0$, $B_1 \subset A_1$ such, that

$0 < \mu(A_1) - \mu(B_1) < \varepsilon$ (6).

Then if we put $B = B_1 \cup \bigcup_{i=2}^n A_i$, the condition (5) is fulfiled.

A_1 is a dyadic element, so there are other dyadic elements A_1^1, $\tilde{A}_1^1$ such, that $A_1^1 \cup \tilde{A}_1^1 = A_1$, $A_1^1 \cap \tilde{A}_1^1 = \emptyset$ and $A_1^1, \tilde{A}_1^1 \neq \emptyset$. Consequently $\mu(A_1) = \mu(A_1^1) + \mu(\tilde{A}_1^1)$. From this is clear, that one of the numbers $\mu(A_1^1)$ and $\mu(\tilde{A}_1^1)$ is not greater then $\frac{1}{2}\mu(A_1)$. Let $\mu(A_1^1)$ have this property. The repetition of the procedure just described will give us for every $k \geqslant 1$ a set $A_1^k \in D$, $\emptyset \subsetneqq A_1^k \subsetneqq A_1$ such, that

$\mu(A_1^k) \le \frac{1}{2^k}\mu(A_1)$ (7).

Since μ is finite we can choose $k_0 \geqslant 1$ such, that $\frac{1}{2^{k_0}}\mu(A_1) < \varepsilon$.

If we put $B_1 = A_1 - A_1^{k_0}$, then $B_1 \in \mathcal{A}_0$ and by (7) there holds that

$\mu(A_1) - \mu(B_1) = \mu(A_1^{k_0}) < \varepsilon$.

The left inequality in (6) can be proved by the following way:

It is enough to show that $\mu(A_1^{k_0}) > 0$. Let us assume, that $\mu(A_1^{k_0})=0$. Thus for every partition $\{C_i\}_{i=1}^{n} \subset \mathcal{A}_1$ of A_1^k there holds, that $\sum_{i=1}^{n} \|M(C_i)\| = 0$. Then the following sequence of implication holds:

$$\|M(C_i)\| = 0 \qquad \text{for every } i=1,2,\dots,n \qquad \Rightarrow$$

$$M(C_i) = \{0\} \qquad \text{for every } i=1,2,\dots,n \qquad \Rightarrow$$

$$M(A_1^{k_0}) = \{0\}.$$

This is a contradiction with the construction of the dyadic structure D_0, so the inequalities in (6) really hold. It means μ is nonatomic.

In [3] is shown, that the total variation of m_j, i.e. $|m_j|$, is the smallest measure with the property (4), thus holds $|m_j|(A) \leq \mu(A)$ for every $A \in \mathcal{A}_1$ and $j=1,2$. This and the nonatomicity of μ conclude the nonatomicity of $|m_j|$ $(j=1,2)$.

From (4) it is obvious, that m_j are μ-continous $(j=1,2)$, thus by the RNP we can find Bochner integrable functions $f_1, f_2 : X \to Y$ such, that $m_j(A) = \int_A f_j \, d\mu$ for $A \in \mathcal{A}_1$ and $j=1,2$.

Let $T=Y \oplus Y$, then T is a Banach space with the RNP. We can easy realise, thet the set function $m: \mathcal{A}_1 \to T$ defined by

$m(A)=(m_1(A), m_2(A))$ for $A \in \mathcal{A}_1$, have the same properties as m_j, i.e. m is a vector measure of nonatomic bounded variation.

We can easy realise, thet the Ljapunoff's theorem for vector measures ([8]) is valid under our conditions, too. Thus holds, that the closure of the domain of m, i.e. the set $K=\mathrm{cl}(\bigcup_{A \in \mathcal{A}_1} m(A))$ is convex. Since $m_j(\emptyset)=0$ and $m_j(X)=x_j$ $(j=1,2)$, holds that $m(\emptyset)=(0,0), m(X)=(x_1,x_2) \in K$. It implies, that

$\alpha(x_1,x_2) \in K$, thus for every $\varepsilon > 0$ there exists

$A \in \mathcal{A}_1$ such that

$$\|\alpha(x_1,x_2) - (m_1(A),m_2(A))\|_T < \frac{\varepsilon}{4} \text{, i.e.}$$

$$\|\alpha x_j - m_j(A)\|_Y < \frac{\varepsilon}{4} \qquad (j=1,2) \tag{8}$$

From the density of $\mathcal{A}_0$ in $\mathcal{A}_1$ in the topology induced by ρ it follows, that there exists a sequence $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}_0$ such that $\mu(A_i \triangle A) \to 0$ $(i \to \infty)$. Since

$$\| m_j(A_i) - m_j(A)\| \leq \int_{A_i \triangle A} \|f_j\| d\mu ,$$

and $\int_E \|f_j\| d\mu (E \in \mathcal{A}_1)$ is absolutly continuous with respect to μ , it is clear, that $\lim_{n \to \infty} \| m_j(A_i) - m_j(A)\| = 0$ $(j=1,2)$. It means there exists $A_0 \in \mathcal{A}_0$ such that $\| m_j(A_0) - m_j(A)\| < \frac{\varepsilon}{4}$. Using this and (8) we receive

$$\|\alpha x_j - m_j(A_0)\| < \frac{\varepsilon}{2} \qquad (j=1,2)$$

We know, that for $A_0 \in \mathcal{A}_0$ holds $m_1(A_0)+m_2(X \setminus A_0) \in M(A_0)+M(X \setminus A_0) = M(X)$ and from (9) follows, that

$$\| \alpha x_1+(1-\alpha)x_2 - (m_1(A_0)+m_2(X \setminus A_0))\| \leq \|\alpha x_1-m_1(A_0)\| + \|\alpha x_2-m_2(A_0)\| < \varepsilon .$$

It means that $\alpha x_1 + (1-\alpha)x_2 \in \mathrm{cl}\, M(X)$.

The proof of Theorem 1 is completed.

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