An Inverstigation on the Operations of the Extended Interval-valued Functions and Their Measurability

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This paper studies some measurability of the extended interval-valued functions and introduce the operations $+, -, \cdot, \div$ of the extended interval-valued functions. Furthermore, We have discussed some measurability of the extended interval-valued functions. On background of the measurability of the extended interval-valued functions, reader may be referred [1,2] ect.

Keywords: Measurable space, Extended interval-valued function, Measurability, Oparation property.

1. Some Measurability of the Extended Interval-valued Functions.

Let $(\Omega, \mathcal{F})$ be a measurable space, where Ω is a fixed nonempty set and $\mathcal{F}$ is a σ field. Let R be the real line, and $\overline{R} = R \cup \{\pm\infty\}$ be the extended real line. Let $(R, \mathcal{B})$ and $(\overline{R}, \overline{\mathcal{B}})$ be the Borel measurable space and the extended Borel measurable space, respectively. Let $\Delta \triangleq \{\langle a,b\rangle : a,b \in \overline{R},\ a \leq b,\ \langle a,b\rangle \neq \phi\}$. Where, $\langle a,b\rangle$ or $[a,b]$, or (a,b), or $[a,b)$ or $(a,b]$. To simplity the writing, set $[a,a] = \{a\} = a$ $(a \in \overline{R})$. If there exist $\varepsilon > 0$, such that $\delta \supset (x-\varepsilon, x+\varepsilon)$, for $\delta \in \Delta$ and $x \in R$, we write "$\delta (\ni) x$". It will be assumed that "$\delta (\ni) +\infty$" $\Longleftrightarrow$ "$\delta \ni +\infty$"; "$\delta (\ni) -\infty$" $\Longleftrightarrow$ "$\delta \ni -\infty$".

Definition 1. $\Gamma : \Omega \dashrightarrow \Delta$ is called "extended interval-valued function.

1) The Γ is called "$\mathcal{F}$-measurable", if for $\forall\ x \in \overline{R}$, "$\Gamma \ni x$" $= \{\omega \in \Omega : \Gamma(\omega) \ni x\} \in \mathcal{F}$.

2) The Γ is called "$\mathcal{F}$-weakly measurable",

if for $\forall$ $x \in \overline{R}$, "$\Gamma (\ni)$ x" $= \{\omega \in \Omega \mid \Gamma(\omega) (\ni) x\} \in \mathcal{F}$.

Definition 2. Let $\Gamma : \Omega \longrightarrow \Delta$. We write E_Γ, $E_\Gamma(\Gamma < x)$, S_Γ, I_Γ that:

$E_\Gamma = \{\omega \in \Omega \mid \Gamma(\omega) = a \ (a \in \overline{R})\}$, $E_\Gamma(\Gamma < x) = \{\omega \in E_\Gamma \mid \Gamma(\omega) < x\}$ $(x \in \overline{R})$,

$S_\Gamma = \{\omega \in \Omega$: there exists $a, b \in R$, $a < b$, such that $\Gamma(\omega) (\ni) a$, $\Gamma(\omega) \ni b$, $\Gamma(\omega) \overline{(\ni)} b\}$,

$I_\Gamma = \{\omega \in \Omega$: there exists $a, b \in R$, $a < b$, such that $\Gamma(\omega) (\ni) b$, $\Gamma(\omega) \ni a$, $\Gamma(\omega) \overline{(\ni)} a\}$,

1) The Γ is called "normal", if $E_\Gamma \in \mathcal{F}$, and $E_\Gamma(\Gamma < x) \in \mathcal{F}$ $(\forall x \in \overline{R})$.

2) The Γ is called "strongly normal", if the Γ is normal and $S_\Gamma \in \mathcal{F}$, $I_\Gamma \in \mathcal{F}$.

Proposition 1. Let $\delta \in \Delta$, and let f, f_1, $f_2 : \Omega \longrightarrow R$ are $\mathcal{F}$-measurable funotions and $f_1 \leq f_2$. We write Γ_i $(i = 1, 2, 3, 4, 5) : \Omega \longrightarrow \Delta$:

$\Gamma_1(\omega) \equiv \delta$, $(\omega \in \Omega)$; $\Gamma_2(\omega) = f(\omega)$, $(\omega \in \Omega)$; $\Gamma_3(\omega) = [f_1(\omega), f_2(\omega)]$, $(\omega \in \Omega)$; $\Gamma_4(\omega) = [f_1(\omega), +\infty)$, $(\omega \in \Omega)$; $\Gamma_5(\omega) = (-\infty, f_2(\omega)]$, $(\omega \in \Omega)$;

then, Γ_i $(i = 1, 2, 3, 4, 5)$ are $\mathcal{F}$-measurable, $\mathcal{F}$-weakly measurable, normal, strongly normal extended interval-valued functions.

Proofs are immediate.

Proposition 2. Let $\Gamma : \Omega \longrightarrow \Delta$; then

Γ is $\mathcal{F}$-measurable $\Longleftrightarrow$ "$\Gamma \supset \delta$" $= \{\omega \in \Omega \mid \Gamma(\omega) \supset \delta\} \in \mathcal{F}$, $(\forall \delta \in \Delta)$.

Proof. C.f. theorem 1.1 of [3].

Proposition 3. Let $\Gamma : \Omega \longrightarrow \Delta$. Let Q be ratinal number set.

If "$\Gamma \ni +\infty$" $\in \mathcal{F}$, "$\Gamma \ni -\infty$" $\in \mathcal{F}$, then

1) Γ is $\mathcal{F}$-weakly measurable $\Longleftrightarrow$ 2) "$\Gamma (\ni) x$" $\in \mathcal{F}$ $(\forall x \in Q)$ $\Longleftrightarrow$ 3) "$\Gamma \supset (r_1, r_2)$" $\in \mathcal{F}$ $(\forall r_1, r_2 \in Q$: $r_1 < r_2)$ $\Longleftrightarrow$ 4) "$\Gamma \supset (a, b)$" $\in \mathcal{F}$ $(\forall a, b \in R$, $a < b)$.

Proof. "1) $\Longrightarrow$ 2)" is immediate.

"2) $\Longrightarrow$ 3)": "$\Gamma \supset (r_1, r_2)$" $= \bigcup_{\substack{h_1, h_2 \in Q \\ r_1 < h_1 < h_2 < r_2}} [$"$\Gamma (\ni) h_1$" $\cap$ "$\Gamma (\ni) h_2$"$] \in \mathcal{F}$.

"3) $\Longrightarrow$ 4)": For $\forall$ $a, b \in R$, $a < b$, there exists $\{r_n\}_{n \geq 1} \subset Q$, $\{h_n\}_{n \geq 1} \subset Q$ such that $r_n < h_n$ $(n \geq 1)$, and $r_n \downarrow a$, $h_n \uparrow b$, it follow by $(a, b) = \bigcup_{n \geq 1} (r_n, h_n)$

that "$\Gamma \supset (a,b)$" $= \bigcup_{n\geq 1}$ "$\Gamma \supset (r_n, h_n)$" $\in \mathcal{F}$.

"4) $\Longrightarrow$ 1)": For $\forall\ x \in R$, there exists $\{a_n\}_{n\geq 1} \subset R$, $\{b_n\}_{n\geq 1} \subset R$, such that $a_n < b_n (n \geq 1)$, $a_n \uparrow x$, $b_n \downarrow x$, it follows that "$\Gamma (\ni)\ x$" $= \bigcup_{n\geq 1}$ "$\Gamma \supset (a_n, b_n)$" $\in \mathcal{F}$. $\square$

Proposition 4. Let $\Gamma : \Omega \longrightarrow \Delta$, and $\overline{\mathcal{B}}$ is the collection of extended Borel sets, then

Γ is normal $\Longleftrightarrow$ $E_\Gamma(\Gamma \in B) \in \mathcal{F}$, $(\forall\ B \in \overline{\mathcal{B}})$.

Proofs are immediate.

Proposition 5. Let $\Gamma : \Omega \longrightarrow \Delta$; then

1) Γ is $\mathcal{F}$-measurable $\Longrightarrow \Gamma$ is $\mathcal{F}$-weakly measurable;

2) Γ is strongly normal $\Longrightarrow$ Γ is normal.

Proofs are immediate.

Proposition6. Let $\Gamma : \Omega \longrightarrow \Delta$, then

Γ is $\mathcal{F}$-weakly measurable and strongly normal $\Longrightarrow \Gamma$ is $\mathcal{F}$-measurable.

Proof. For $\forall\ x \in R$, by the conditions, we have that "$\Gamma \ni x$" $\cap$ "$\Gamma \overline{(\ni)} x$" $=$

$$E_\Gamma(\Gamma = x) \cup \left\{\left[\bigcup_{\substack{r_1 \in Q \\ r_1 < x}} \left(\bigcap_{\substack{r \in Q \\ r_1 < r < x}} \text{"}\Gamma (\ni) r\text{"}\right)\right] \cap \left[\bigcap_{\substack{r_2 \in Q \\ r_2 > x}} \text{"}\Gamma \overline{(\ni)} r_2\text{"}\right] \cap S_\Gamma\right\} \cup \left\{\left[\bigcup_{\substack{r_2 \in Q \\ r_2 > x}} \left(\bigcap_{\substack{r \in Q \\ r_2 > r > x}} \text{"}\Gamma (\ni) r\text{"}\right)\right] \cap \left[\bigcap_{\substack{r_1 \in Q \\ r_1 < x}} \text{"}\Gamma \overline{(\ni)} r_1\text{"}\right] \cap I_\Gamma\right\} \in \mathcal{F}.$$

Consequently, "$\Gamma \ni x$" $=$ "$\Gamma (\ni)\ x$" $\cup$ ["$\Gamma \ni x$" $\cap$ "$\Gamma \overline{(\ni)}\ x$" $\in \mathcal{F}$. $\square$

Remark 1. The following example 1 states necessity of the condition "strongly normal" in the proposition 6.

Example 1. Let $\Omega = R$, $(R, \mathcal{F})$ be Lebesgue measurable space, $E \subset (0,1)$ and $E \overline{\in} \mathcal{F}$. Let $\Gamma : \Omega \longrightarrow \Delta$:

$$\Gamma(\omega) = \begin{cases} (-\infty, 0], & \text{if } \omega \in E, \\ (-\infty, 0), & \text{if } \omega \in R - E. \end{cases}$$

Clealy, 1) Γ is $\mathcal{F}$-weakly measurable and normal, but Γ is not strongly normal

2) Since "$\Gamma \ni 0$" $= E \overline{\in} \mathcal{F}$, and hence Γ is not $\mathcal{F}$-measurable.

2. The operrations of extended interval-valued functions and their measurability

Definition 3. Let $* \in \{+, -, \cdot, \div\}$. The expressions $0\cdot(\pm\infty)$, $(\pm\infty)-(\pm\infty)$, $\frac{\pm\infty}{\pm\infty}$, $\frac{a}{0}$ $(a \in \bar{R})$ are meaningless.

1) For $\delta_1, \delta_2 \in \Delta$, we have difined $\delta_1 * \delta_2 = \{x = x_1 * x_2 : x_1 \in \delta_1, x_2 \in \delta_2\}$. Where, we assume that $x_1 * x_2$ exists for $\forall x_1 \in \delta_1$ and $x_2 \in \delta_2$.

2) For $\Gamma_1, \Gamma_2 : \Omega \longrightarrow \Delta$, we defined $\Gamma_1 * \Gamma_2 : \Omega \longrightarrow \Delta$:

$(\Gamma_1 * \Gamma_2)(\omega) = \Gamma_1(\omega) * \Gamma_2(\omega)$, $(\omega \in \Omega)$.

Where, we assume that $\Gamma_1(\omega) * \Gamma_2(\omega)$ exists for $\forall \omega \in \Omega$.

Property 1. Let $\Gamma : \Omega \longrightarrow \Delta$, and $a, b \in R$, then

1) Γ is $\mathcal{F}$-measurable $\Longrightarrow$ $a + b\cdot\Gamma$ is $\mathcal{F}$-measurable;

2) Γ is $\mathcal{F}$-weakly measurable $\Longrightarrow$ $a + b\cdot\Gamma$ is $\mathcal{F}$-weakly measurable.

Proof. The assertuons follows from the fact for $\forall x \in R$ that

$$\text{“}a+b\cdot\Gamma \ni x\text{”} = \begin{cases} \Omega, & \text{if } b=0,\ a=x, \\ \phi, & \text{if } b=0,\ a\neq x, \\ \text{“}\Gamma \ni \frac{x-a}{b}\text{”}, & \text{if } b\neq 0; \end{cases} \qquad \text{“}a+b\cdot\Gamma\,(\ni)\,x\text{”} = \begin{cases} \phi, & \text{if } b=0, \\ \text{“}\Gamma\,(\ni)\,\frac{x-a}{b}\text{”}, & \text{if } b\neq 0; \end{cases}$$

and

$$\text{“}a+b\cdot\Gamma \ni +\infty\text{”} = \begin{cases} \phi, & \text{if } b=0, \\ \text{“}\Gamma\,(\ni)\,+\infty\text{”}, & \text{if } b>0, \\ \text{“}\Gamma\,(\ni)\,-\infty\text{”}, & \text{if } b<0; \end{cases} \qquad \text{“}a+b\cdot\Gamma \ni -\infty\text{”} = \begin{cases} \phi, & \text{if } b=0, \\ \text{“}\Gamma \ni -\infty\text{”}, & \text{if } b>0, \\ \text{“}\Gamma \ni +\infty\text{”}, & \text{if } b<0. \end{cases} \qquad \square$$

Lemma 1. Let $\Gamma : \Omega \longrightarrow \Delta$; then

1) Γ_1, Γ_2 are normal $\Longrightarrow$ $\Gamma_1 + \Gamma_2$ is normal;

2) Γ_1, Γ_2 are strongly normal $\Longrightarrow$ $\Gamma_1 + \Gamma_2$ is strongly normal.

Proof. 1) The assertions follows from the fact that $E_{\Gamma_1+\Gamma_2} = E_{\Gamma_1} \cap E_{\Gamma_2}$;

$$E_{\Gamma_1+\Gamma_2}(\Gamma_1+\Gamma_2 < x) = E_{\Gamma_1+\Gamma_2}(\Gamma_1 < x - \Gamma_2) = \bigcup_{r\in Q} E_{\Gamma_1+\Gamma_2}(\Gamma_1 < r < x - \Gamma_2) = \bigcup_{r\in Q} \left[E_{\Gamma_1}(\Gamma_1 < r) \cap E_{\Gamma_2} \cap E_{\Gamma_2}(\Gamma_2 < x - r) \cap E_{\Gamma_1}\right], (\forall x \in R);$$

$$E_{\Gamma_1+\Gamma_2}(\Gamma_1+\Gamma_2 < +\infty) = E_{\Gamma_1}(\Gamma_1 < +\infty) \cap E_{\Gamma_2}(\Gamma_2 < +\infty), \quad E_{\Gamma_1+\Gamma_2}(\Gamma_1+\Gamma_2 < -\infty) = \phi.$$

2) The assertions follows from 1) and fact that

Property 2. Let $\Gamma_1, \Gamma_2 : \Omega \longrightarrow \Delta$. Then

1) Γ_1, Γ_2 are $\mathcal{F}$-weakly measurable and normal $\Longrightarrow$ $\Gamma_1 + \Gamma_2$ is $\mathcal{F}$-weakly measurable and normal;

2) Γ_1, Γ_2 are $\mathcal{F}$-weakly measurable and strongly normal $\Longrightarrow$ $\Gamma_1 + \Gamma_2$ is $\mathcal{F}$-measurable and strongly normal.

Proof. 1) The assertions follows from lemma 1, proposition 4 and fact that

$$\text{“}\Gamma_1+\Gamma_2 (\ni)\, x\text{”} = \bigcup_{n\geq 1} \bigcup_{\substack{\gamma_1,\gamma_2 \in Q \\ 0<\gamma_2-\gamma_1<\frac{1}{n}}} \Big\{ \Big[\text{“}\Gamma_1 \supset (\gamma_1,\gamma_2)\text{”} \cap \text{“}\Gamma_2 \supset (x-\gamma_1-\tfrac{1}{n},\, x-\gamma_2+\tfrac{1}{n})\text{”}\Big] \cup \Big[\text{“}\Gamma_2 \supset (\gamma_1,\gamma_2)\text{”} \cap \text{“}\Gamma_1 \supset (x-\gamma_1-\tfrac{1}{n},$$
$$x-\gamma_2+\tfrac{1}{n})\text{”}\Big] \cup \Big[\text{“}\Gamma_1 \supset (\gamma_1-\tfrac{1}{n}, \gamma_2+\tfrac{1}{n})\text{”} \cap E_{\Gamma_2}(x-\gamma_1-\tfrac{1}{n} < \Gamma_2 < x-\gamma_2+\tfrac{1}{n})\Big] \cup \Big[\text{“}\Gamma_2 \supset (\gamma_1-\tfrac{1}{n}, \gamma_2+\tfrac{1}{n})\text{”} \cap E_{\Gamma_1}(x-\gamma_1-\tfrac{1}{n},\, x-\gamma_2+\tfrac{1}{n})\text{”}\Big]\Big\}$$

$(\forall x \in R)$;

$$\text{“}\Gamma_1+\Gamma_2 \ni +\infty\text{”} = \text{“}\Gamma_1 \ni +\infty\text{”} \cup \text{“}\Gamma_2 \ni +\infty\text{”}, \qquad \text{“}\Gamma_1+\Gamma_2 \ni -\infty\text{”} = \text{“}\Gamma_1 \ni -\infty\text{”} \cup \text{“}\Gamma_2 \ni -\infty\text{”}.$$

2) The assertions follows from 1), lemma 1 and proposition 6.

Lemma 2. Let : — be strongly normal and -weakly measurable. We write

$$S_{+\infty} = \bigcup_{n\geq 1}\Big[\bigcap_{K\geq n} \text{“}\Gamma (\ni) K\text{”}\Big] \cup E_\Gamma(\Gamma = +\infty), \qquad I_{-\infty} = \bigcup_{n\geq 1}\Big[\bigcap_{K\geq n} \text{“}\Gamma (\ni) -K\text{”}\Big] \cup E_\Gamma(\Gamma = -\infty),$$

$$\Omega_S = \Omega - [S_\Gamma \cup E_\Gamma \cup S_{+\infty}], \qquad \Omega_I = \Omega - [I_\Gamma \cup E_\Gamma \cup I_{-\infty}].$$

$\Gamma_{S_0}, \Gamma_{I_0}, \Gamma_{(S_0)}, \Gamma_{(I_0)} : \Omega \longrightarrow R$ defined by that

$$\Gamma_{S_0}(\omega) = \begin{cases} \max\limits_{\Gamma(\omega)\ni x}\{x\}, & \text{if } \omega \in S_\Gamma, \\ 0, & \text{if } \omega \bar{\in} S_\Gamma; \end{cases} \qquad \Gamma_{I_0}(\omega) = \begin{cases} \min\limits_{\Gamma(\omega)\ni x}\{x\}, & \text{if } \omega \in I_\Gamma, \\ 0, & \text{if } \omega \bar{\in} I_\Gamma; \end{cases}$$

$$\Gamma_{(S_0)}(\omega) = \begin{cases} \sup\limits_{\Gamma(\omega)\ni x}\{x\}, & \text{if } \omega \in \Omega_S, \\ 0, & \text{if } \omega \bar{\in} \Omega_S; \end{cases} \qquad \Gamma_{(I_0)}(\omega) = \begin{cases} \inf\limits_{\Gamma(\omega)\ni x}\{x\}, & \text{if } \omega \in \Omega_I, \\ 0, & \text{if } \omega \bar{\in} \Omega_I. \end{cases}$$

Then $\Gamma_{S_0}, \Gamma_{I_0}, \Gamma_{(S_0)}, \Gamma_{(I_0)} : \Omega \longrightarrow \bar{R}$ are $\mathcal{F}-\mathcal{B}$ measurable functions.

Proof. For $\forall$ c $\in$R, we have

$$\text{"}\Gamma_{S_0} \le c\text{"} = (\Omega - S_\Gamma) \cup \Big\{ S_\Gamma \cap \Big(\bigcup_{r_1 \in Q} \text{"}\Gamma(\ni) r_1\text{"}\Big) \cap \Big(\bigcap_{\substack{r_2 \in Q \\ r_2 > c}} \text{"}\Gamma \overline{(\ni)} r_2\text{"}\Big)\Big\}, \ (\text{if } c \ge 0);$$

$$\text{"}\Gamma_{S_0} \le c\text{"} = S_\Gamma \cap \Big(\bigcup_{r_1 \in Q} \text{"}\Gamma(\ni) r_1\text{"}\Big) \cap \Big(\bigcap_{\substack{r_2 \in Q \\ r_2 > c}} \text{"}\Gamma(\ni) c\text{"}\Big), \ (\text{if } c < 0);$$

$$\text{"}\Gamma_{(I_0)} \ge c\text{"} = (\Omega - \Omega_I) \cup \Big\{ \Omega_I \cap \Big(\bigcup_{r_1 \in Q} \text{"}\Gamma(\ni) r_1\text{"}\Big) \cap \Big(\bigcap_{\substack{r_2 \in Q \\ r_2 < c}} \text{"}\Gamma \overline{(\ni)} r_2\text{"}\Big), \ (\text{if } c \le 0);$$

$$\text{"}\Gamma_{(I_0)} \ge c\text{"} = \Omega_I \cap \Big(\bigcup_{r_1 \in Q} \text{"}\Gamma(\ni) r_1\text{"}\Big) \cap \Big(\bigcap_{\substack{r_2 \in Q \\ r_2 < c}} \text{"}\Gamma \overline{(\ni)} r_2\text{"}\Big), \ (\text{if } c > 0)$$

It follows, by the conditions, that

$$\text{"}\Gamma_{S_0} \le c\text{"} \in \mathcal{F}, \quad \text{"}\Gamma_{(I_0)} \ge c\text{"} \in \mathcal{F} \quad (\forall c \in R).$$

Thus, Γ_{S_0} and $\Gamma_{(I_0)}$ are $\mathcal{F}$-$\mathcal{B}$ measurable functions.

$\mathcal{F}$-$\mathcal{B}$ measurability of Γ_{I_0} and $\Gamma_{(S_0)}$ are similarly proved.

Lemma 3. Let $\Gamma : \Omega \longrightarrow \Delta$. For $\delta \in \Delta$, we write $\delta^{(2)} = \{y = x^2 : x \in \delta\}$ and let $\Gamma^{(2)} : \Omega \longrightarrow \Delta : \ \Gamma^{(2)}(\omega) = (\Gamma(\omega))^{(2)} \ (\omega \in \Omega)$. We have

1) Γ is normal and $\mathcal{F}$-weakly measurable $\Longrightarrow \Gamma^{(2)}$ is normal and $\mathcal{F}$-weakly measurable ;

2) Γ is strongly normal and $\mathcal{F}$-weakly measurable $\Longrightarrow \Gamma^{(2)}$ is strongly normal.

Proof. 1) The normality of $\Gamma^{(2)}$ follows from the fact that $E_{\Gamma^{(2)}} = E_\Gamma$, and for $\forall x \in \overline{R}$

$$E_{\Gamma^{(2)}}(\Gamma^{(2)} < x) = \begin{cases} E_\Gamma(\Gamma < \sqrt{x}) \cup E_\Gamma(\Gamma > -\sqrt{x}), & \text{if } x > 0, \\ \phi, & \text{if } x \le 0. \end{cases}$$

The $\mathcal{F}$-weakly measurability of $\Gamma^{(2)}$ follows from the fact for $\forall x \in R$ that

$$\text{"}\Gamma^{(2)}(\ni) x\text{"} = \begin{cases} \text{"}\Gamma(\ni)\sqrt{x}\text{"} \cup \text{"}\Gamma(\ni) -\sqrt{x}\text{"}, & \text{if } x > 0, \\ \phi, & \text{if } x \le 0. \end{cases}$$

2) It follows readily that

$$S_{\Gamma^{(2)}} = \{\text{"}|\Gamma_{S_0}| > |\Gamma_{I_0}|\text{"} \cup \text{"}|\Gamma_{I_0}| > |\Gamma_{S_0}|\text{"} \cup \text{"}|\Gamma_{S_0}| > |\Gamma_{(I_0)}|\text{"} \cup \text{"}|\Gamma_{I_0}| > |\Gamma_{(S_0)}|\text{"}\} - I_{-\infty};$$

$$I_{\Gamma^{(2)}} = \{\text{"}|\Gamma_{S_0}| < |\Gamma_{I_0}|\text{"} \cup \text{"}|\Gamma_{I_0}| < |\Gamma_{S_0}|\text{"} \cup \text{"}|\Gamma_{S_0}| < |\Gamma_{(I_0)}|\text{"} \cup \text{"}|\Gamma_{I_0}| < |\Gamma_{(S_0)}|\text{"}\} - S_{+\infty}.$$

Thus, by the conditions and lemma 2, $S_{\Gamma^{(2)}}, I_{\Gamma^{(2)}} \in \mathcal{F}$. Therefore, by 1), $\Gamma^{(2)}$ is strongly normal.

Property 3. Let $\Gamma_1, \Gamma_2 : \Omega \longrightarrow \Delta$; then

1) Γ_1 and Γ_2 are normal and $\mathcal{F}$-weakly normal $\Longrightarrow$ $\Gamma_1 \cdot \Gamma_2$ is normal and $\mathcal{F}$-weakly measurable;

2) Γ_1 and Γ_2 are strongly normal and $\mathcal{F}$-weakly measurable $\Longrightarrow$ $\Gamma_1 \cdot \Gamma_2$ is strongly normal and $\mathcal{F}$-measurable.

Proof. It follows readily that $\Gamma_1 \cdot \Gamma_2 = \frac{1}{2}[(\Gamma_1 + \Gamma_2)^{(2)} - (\Gamma_1^{(2)} + \Gamma_2^{(2)})]$.

The assertion 1) follows at once from lemma 3 and property 2.

The assertion 2) follows at once from lemma 3 and property 6.

Lemma 4. Let $\Gamma : \Omega \longrightarrow \Delta$, and let $\Gamma^{-1} = \frac{1}{\Gamma} = 1 \div \Gamma : \Omega \longrightarrow \Delta$. We have

1) Γ is normal $\Longrightarrow$ Γ^{-1} is normal;

2) Γ is $\mathcal{F}$-weakly measurable $\Longrightarrow$ Γ^{-1} is $\mathcal{F}$-weakly measurable;

3) Γ is strongly normal $\Longrightarrow$ Γ^{-1} is strongly normal.

Proof. 1) It follows readily that $E_{\Gamma^{-1}} = E_{\Gamma}$, and for $\forall c \in R$,

$$E_{\Gamma^{-1}}(\Gamma^{-1} < c) = \begin{cases} E_\Gamma(\Gamma < 0) \cup E_\Gamma(\Gamma > \frac{1}{c}), & \text{if } c > 0, \\ E_\Gamma(\Gamma < 0), & \text{if } c = 0, \\ E_\Gamma(\Gamma < 0) \cup E_\Gamma(\Gamma > \frac{1}{c}), & \text{if } c < 0. \end{cases}$$

Thus, by the conditions, Γ^{-1} is normal.

2) The assertion follows from condition and fact for $\forall x \in \overline{R}$ that

$$``\Gamma^{-1}(\ni) x" = \begin{cases} ``\Gamma(\ni)\frac{1}{x}", & \text{if } x \in R - \{0\}, \\ \phi, & \text{if } x = 0. \end{cases}$$

3) The assertion follows from 1) and fact that $S_{\Gamma^{-1}} = I_\Gamma$, $I_{\Gamma^{-1}} = S_\Gamma$. □

Property 4. Let $\Gamma_1, \Gamma_2 : \Omega \longrightarrow \Delta$, then

1) Γ_1, Γ_2 are normal and $\mathcal{F}$-weakly measurable $\Longrightarrow$ $\Gamma_1 \div \Gamma_2$ is normal and $\mathcal{F}$-weakly measurable;

2) Γ_1, Γ_2 are strongly normal and $\mathcal{F}$-weakly measurable $\Longrightarrow$ $\Gamma_1 \div \Gamma_2$ is strongly normal and $\mathcal{F}$-measurable.

Poof. The assertions follows at once from lemma 4 and property 4.

Remark 2. The follows example 2 state that necessity of the condition "Γ_1, Γ_2 are normal" for $\mathcal{F}$-weakly measurability of $\Gamma_1 * \Gamma_2$ in the results of properties 2,3,4.

Example 2. Let $\Gamma_1, \Gamma_2 : \Omega \longrightarrow \Delta$:

$$\Gamma_1(\omega) \equiv [1,3], \ (\omega \in \Omega); \qquad \Gamma_2(\omega) = \begin{cases} \omega+3, & \text{if } \omega \in E, \\ \omega, & \text{if } \omega \in (0,1)-E, \\ [-5,-4], & \text{if } \omega \in R-(0,1); \end{cases} \qquad \Gamma_3 = \Gamma_2^{-1}.$$

Where, $\Omega = R$, $E \subset (0,1)$, $E \bar{\in} \mathcal{F}$ (c.f. example 1).

Clealy, 1) Γ_1 is strongly normal and $\mathcal{F}$-measurable; Γ_2 is $\mathcal{F}$-measurable, but is not normal; 2) since "$\Gamma_1 + \Gamma_2 \ (\ni) \ 5.5$" = "$\Gamma_1 \cdot \Gamma_2 \ (\ni) \ 6$" = $E \bar{\in} \mathcal{F}$; thus, $\Gamma_1 * \Gamma_2$ is not $\mathcal{F}$-weakly measurable.

= "$\Gamma_1 - (-\Gamma_2) \ (\ni) \ 5.5$"
= "$\Gamma_1 \div \Gamma_3$"

Where, $* \in \{ +, -, \ , - \}$.

The follows example 3 state necessity of the conditions "Γ_1, Γ_2 are strongly normal" for $\mathcal{F}$-measurability of $\Gamma_1 * \Gamma_2$ in the results of properties 2, 3, 4.

Example 3. Let Γ_i ($i = 1, \cdots, 5$) $: \Omega \longrightarrow \Delta$;

$$\Gamma_1(\omega) = \begin{cases} (-\infty, \omega], & \text{if } \omega \in E, \\ (-\infty, \omega), & \text{if } \omega \in (0,1)-E, \\ [4,5), & \text{if } \omega \in R-(0,1); \end{cases} \qquad \Gamma_2(\omega) = \begin{cases} (-\infty, 1-\omega], & \text{if } \omega \in E, \\ (-\infty, 1-\omega), & \text{if } \omega \in (0,1)-E, \\ [4,5), & \text{if } \omega \in R-(0,1); \end{cases}$$

$$\Gamma_3(\omega) = \begin{cases} [\omega, +\infty), & \text{if } \omega \in E, \\ (\omega, +\infty), & \text{if } \omega \in (0,1)-E, \\ (-2,-1], & \text{if } \omega \in R-(0,1); \end{cases} \qquad \Gamma_4(\omega) = \begin{cases} [\frac{3}{\omega}, +\infty), & \text{if } \omega \in E, \\ (\frac{3}{\omega}, +\infty), & \text{if } \omega \in (0,1)-E, \\ [-1,-1], & \text{if } \omega \in R-(0,1); \end{cases}$$

$$\Gamma_5 = \Gamma_4^{-1}.$$

It follows readily that $\Gamma_1 + \Gamma_2, \ \Gamma_3 \cdot \Gamma_4, \ \Gamma_3 \div \Gamma_5 : \Omega \longrightarrow \Delta$:

$$(\Gamma_1 + \Gamma_2)(\omega) = \begin{cases} (-\infty, 1], & \text{if } \omega \in E, \\ (-\infty, 1), & \text{if } \omega \in (0,1)-E, \\ [8,10), & \text{if } \omega \in R-(0,1); \end{cases} \qquad (\Gamma_3 \cdot \Gamma_4)(\omega) = (\Gamma_3 \div \Gamma_5)(\omega) = \begin{cases} [3,+\infty), & \text{if } \omega \in E, \\ (3,+\infty), & \text{if } \omega \in (0,1)-E, \\ [1,2), & \text{if } \omega \in R-(0,1). \end{cases}$$

Clealy, 1) Γ_i (i=1, ···,5) are normal and $\mathcal{F}$-measurable, but Γ_2, Γ_3, Γ_5 are not strongly normal; 2) since "$\Gamma_1 + \Gamma_2 \ni 1$" = "$\Gamma_3 \cdot \Gamma_4 \ni 3$" = "$\Gamma_3 \div \Gamma_5 \ni 3$" = $E \bar{\in} \mathcal{F}$, hence $\Gamma_1 + \Gamma_2$, $\Gamma_3 \cdot \Gamma_4$, $\Gamma_3 \div \Gamma_5$ are not $\mathcal{F}$-measurable.

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