$\wedge$-pseudometrics as families of ordinary pseudometrics

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The aim of this paper is to show that probabilistic pseudometrics under $\wedge$ ($\wedge$-pseudometrics) and $]0,1[$-indexed families $(m_\alpha)_{\alpha\in]0,1[}$ of ordinary pseudometrics m_α provided with the property

$$m_\alpha = \inf_{\beta\in]0,\alpha[} m_\beta \quad \text{for every } \alpha\in]0,1[$$

are equivalent notions.

Let us first recall some definitions:

$\mathcal{D}^+$ denotes the set of all nonnegative probability distribution functions - i.e. all increasing, left continuous functions $F:\mathbf{R} \to [0,1]$ with $F(0)=0$ and $\sup_{x\in\mathbf{R}} F(x) = 1$.

$\mathcal{D}^+$ is partially ordered by

$$F,G\in \mathcal{D}^+: \quad F\le G \quad \Leftrightarrow \quad F(x) \ge G(x) \quad \text{for every } x\in\mathbf{R}$$

and with the binary operation $\tau_\wedge$ defined by

$$F,G\in \mathcal{D}^+; x\in\mathbf{R}: \quad \tau_\wedge(F,G)(x) := \sup_{\substack{y,z\in\mathbf{R}\\ y+z=x}} F(y)\wedge G(z)$$

$(\mathcal{D}^+,\tau_\wedge)$ is a semigroup with unit element ϵ_0 given by

$$x\in\mathbf{R}: \quad \epsilon_0(x) := \begin{cases} 1, & 0<x \\ 0, & 0\ge x \end{cases} .$$

S denotes a nonvoid set.

Definition: (cf. [5]) A mapping $m:S\times S \to \mathcal{D}^+$ is said to be a probabilistic pseudometric under $\wedge$ ($\wedge$-pseudometric) on S iff it satisfies the conditions

(M1) $\quad m(p,p) = \epsilon_0 \quad$ for every $p\in S$

(M2) $\quad m(p,q) = m(q,p) \quad$ for every $p,q\in S$

(M3) $\quad m(p,q) \le \tau_\wedge(m(p,r),m(r,q)) \quad$ for every $p,q,r\in S$.

Let m be a $\wedge$-pseudometric on S and for every $\alpha\in]0,1[$ let $m_\alpha : S\times S \to \mathbf{R}^+$ be defined by:

$$(1) \quad (p,q)\in S\times S:\ m_\alpha(p,q) := \inf_{\beta\in]1-\alpha,1[} \sup\{x\in\mathbf{R} \mid m(p,q)(x)<\beta\} .$$

Lemma: $(m_\alpha)_{\alpha\in]0,1[}$ is a family of ordinary pseudometrics m_α on S provided with the property

$$(\mathrm{MC}) \quad m_\alpha(p,q) = \inf_{\beta\in]0,\alpha[} m_\beta(p,q) \qquad \text{for every } \alpha\in]0,1[;\ p,q\in S .$$

Moreover, the following relation holds for every $p,q\in S$; $x\in\mathbf{R}$:

$$(2) \quad m(p,q)(x) = \sup\{\alpha\in]0,1[\mid m_{1-\alpha}(p,q)<x\} .$$

Proof: (M1) implies $m_\alpha(p,p)=0$ for every $p,q\in S$; $\alpha\in]0,1[$ and from (M2) we infer $m_\alpha(p,q)=m_\alpha(q,p)$ for every $p,q\in S$; $\alpha\in]0,1[$.
To prove the triangle inequality assume the existence of $p,q,r\in S$ and $0<y\in\mathbf{R}$ with

$$m_\alpha(p,r)-\frac{y}{2} \geq m_\alpha(p,q)+m_\alpha(q,r)+\frac{y}{2} .$$

Then

$$m(p,r)[m_\alpha(p,r)-\frac{y}{2}] \leq 1-\alpha .$$

Since for some $\beta,\gamma\in]1-\alpha,1[$

$$m(p,q)[m_\alpha(p,q)+\frac{y}{4}] \geq \beta \quad \text{and} \quad m(q,r)[m_\alpha(q,r)+\frac{y}{4}] \geq \gamma$$

we obtain in view of (M3) the contradiction

$$\beta\wedge\gamma \leq m(p,q)[m_\alpha(p,q)+\frac{y}{4}] \wedge m(q,r)[m_\alpha(q,r)+\frac{y}{4}] \leq$$

$$\leq m(p,r)[m_\alpha(p,q)+m_\alpha(q,r)+\frac{y}{2}] \leq m(p,r)[m_\alpha(p,r)-\frac{y}{2}] \leq 1-\alpha .$$

(MC) follows from the definition.
To prove (2) note that in view of the definitions the following equivalence holds true for every $p,q\in S$; $\alpha\in]0,1[$ and $x\in\mathbf{R}$:

$$(*) \quad m(p,q)(x) > \alpha \quad \Leftrightarrow \quad m_{1-\alpha}(p,q) < x .$$

Proposition: The relation

$$m \mapsto (m_\alpha)_{\alpha\in]0,1[}$$

defines a bijective mapping Ω_S from the set of all $\wedge$-pseudometrics m on S onto the set of all $]0,1[$-indexed families $(m_\alpha)_{\alpha\in]0,1[}$ of ordinary pseudometrics m_α on S provided with the property (MC).

Proof: The injectivity of Ω_S follows immediatly from (2).
To prove that Ω_S is surjective let $(m_\alpha)_{\alpha\in]0,1[}$ be a family of pseudometrics m_α on S possessing the property (MC) and define

$m: S\times S \to [0,1]^{\mathbf{R}}$ according to (2).
Then in view of (MC) the following relation holds true for every $p,q\in S$; $\alpha\in]0,1[$ and $x\in\mathbf{R}$:

$$(*) \quad m(p,q)(x) > \alpha \iff m_{1-\alpha}(p,q) < x \quad .$$

(i) $m(p,q)\in \mathcal{D}^+$ for every $p,q\in S$ follows immediately from (*).
(ii) m is a $\wedge$-pseudometric:
(M1) and (M2) hold true since m_α is a pseudometric for all $\alpha\in]0,1[$.
To prove (M3) assume the existence of $p,q,r\in S$; $x,y\in\mathbf{R}$ and $\alpha\in]0,1[$ with

$$m(p,q)(x+y) < 1-\alpha < m(p,r)(x) \wedge m(r,q)(y) \; .$$

Then we infer from (*) the contradiction

$$x+y \le m_\alpha(p,q) \le m_\alpha(p,r) + m_\alpha(r,q) < x+y \; .$$

(iii) Finally, the relation

$$m_\alpha(p,q) = \inf_{\beta\in]1-\alpha,1[} \sup\{x\in\mathbf{R} \mid m(p,q)(x) < \beta\} \quad \text{for every } p,q\in S;\ \alpha\in]0,1[$$

follows immediately from (*).

According to this Proposition $\wedge$-pseudometrics and $]0,1[$-indexed families of ordinary pseudometrics provided with the property (MC) are equivalent notions.
In particular, this Proposition shows that pseudometrizable fuzzy neighborhood spaces (S,m) and $]0,1[$-indexed families $((S,m_\alpha))_{\alpha\in]0,1[}$ of ordinary pseudometrizable spaces (S,m_α) possessing the property (MC) can be identified in the sense of (1) or (2).
Moreover, in view of Proposition 1 in [2] and Corollary 3.2.2 in [1] this identification preserves topologies; i.e.:

(i) if (S,Δ) is a pseudometrizable fuzzy neighborhood space with generating $\wedge$-pseudometric m on S and $(m_\alpha)_{\alpha\in]0,1[} := \Omega_S(m)$ then for every $\alpha\in]0,1[$ the α-level-topology of (S,Δ) is generated by the ordinary pseudometric m_α on S and
(ii) if (S,Δ) is a fuzzy neighborhood space and $(m_\alpha)_{\alpha\in]0,1[}$ is a $]0,1[$-indexed family of ordinary pseudometrics m_α on S provided with property (MC) so that for every $\alpha\in]0,1[$ the α-level-topology of (S,Δ) is generated by m_α then (S,Δ) is generated by $\Omega_S^{-1}((m_\alpha)_{\alpha\in]0,1[})$.

References

[1] W. Flüshöh: On the representation of fuzzy neighborhood spaces, Comm. IFSA Math. Chapter Vol. 2 (1988) 14-19.

[2] W. Flüshöh: A characterization of metrizable fuzzy neighborhood spaces, submitted to Comm. IFSA Math. Chapter.

[3] U.Höhle: Probabilistic metrization of fuzzy uniformities, Fuzzy Sets and Systems 8 (1982) 63-69.

[4] R. Lowen: Fuzzy neighborhood spaces, Fuzzy Sets and Systems 7 (1982) 165-189.

[5] B. Schweizer and A. Sklar: Probabilistic metric spaces, North-Holland 1983.