THE LATTICE OF THE NECESSITIES

by

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1. Introduction.

This paper follows Biacino and Gerla [1990]b. In Section 2 we give further information about the generation of the necessities and the possibilities. In particular, we relate the necessities generated by a finite initial valuation with Shafer's consonant belief functions. In Section 3 the lattice of the necessities is examined. In Section 4 the formulas we propose in Biacino and Gerla [1990]b for the necessities and possibilities generated by an initial valuation are used to define conditional necessities and possibilities. The notion we arrive at is very close to what D.Dubois and H.Prade have proposed in D.Dubois and H.Prade [1988].

2. Generation of necessities and possibilities: further information.

We say that an initial valuation is *consonant* if D_f is a finite chain $e_1 < e_2 < \ldots < e_k$, and $f(e_1) \leq f(e_2) \leq \ldots \leq f(e_k)=1$.

Proposition 2.1 For every $n \in F(B)$ the following are equivalent:

a) n is a necessity generated by a finite initial valuation ;

b) n is generated by a consonant initial valuation.

The necessities generated by a consonant valuation are related to the Shafer's consonant belief functions.

Proposition 2.2. n is a completely consistent necessity generated by a finite initial valuation if and only if n is a consonant belief function.

If B is a finite Boolean algebra, then every necessity is finitely generated and therefore the completely consistent necessities coincide with the consonant belief functions. On the contrary, if B is infinite, then there exists a completely consistent necessity that is not a consonant belief function. Indeed, in this case a non principal filter exists in B and its characteristic function is a completely consistent necessity that is not generated by a finite initial valuation.

Proposition 2.3 . The necessity $\bar{f}$ is completely consistent if and only if every finite meet of elements of $\{x \in D_f / f(x) \neq 0\}$ is different from 0. The possibility $\mathring{f}$ is completely consistent if and only if every finite join of elements of $\{x \in D_f / f(x) \neq 1\}$ is different from 1.

The following proposition shows that the operators $^{-}$ and $^{\circ}$ have properties very like to the closure and interior operators in topological space theory.

Proposition 2.4 The following hold:

(2.1) f necessity $\Longleftrightarrow f = \bar{f}$;

(2.2) f possibility $\Longleftrightarrow f = \mathring{f}$;

(2.3) $\mathring{f}(x) \leq f(x) \leq \bar{f}(x)$ for every $x \in D_f$;

(2.4) $\sim\bar{f} = (\sim f)^{\circ}$; $\sim\mathring{f} = (\sim f)^{-}$; $(\bar{f})^{-} = \bar{f}$; $(\mathring{f})^{\circ} = \mathring{f}$;

(2.5) $f \leq g \Rightarrow \bar{f} \leq \bar{g}$; $f \leq g \Rightarrow \mathring{f} \leq \mathring{g}$.

Proposition 2.5 Given an initial valuation f, we have

(2.6) $C_r(f) + C_s(\sim f) = 1$;

(2.7) $f \geq \sim f \Rightarrow 1 \leq \bar{f}(x) + \bar{f}(-x) \leq 1 + C_r(f)$;

(2.8) $f \leq \sim f \Rightarrow C_s(f) \leq \mathring{f}(x) + \mathring{f}(-x) \leq 1$;

(2.9) $f \geq \sim f$ and $C_r(f) = 0 \Rightarrow \bar{f} = \sim\bar{f}$ and $\bar{f}$ prime filter ;

(2.10) $f \leq \sim f$ and $C_s(f) = 1 \Rightarrow \mathring{f} = \sim\mathring{f}$ and $\mathring{f}$ prime filter .

We say that an initial valuation f is *balanced* provided that $\sim f = f$, i.e. $x \in D_f$ implies $-x \in D_f$ and $f(x) + f(-x) = 1$ for every $x \in D_f$. If the initial valuation is balanced then $\mathring{f} = \sim\bar{f}$ and $\bar{f}$ is completely consistent if and only if $\bar{f}$ is a Boolean valuation. An initial valuation $f:B\to[0,1]$ is called *n-stable* if $\bar{f}$ is an extension of f, i.e. $\bar{f}(x)=f(x)$ for every $x\in D_f$; f is called *p-stable* if $\mathring{f}$ is an extension of f.

The following proposition characterizes the p-stable and the n-stable initial valuations (see also Zhang Guangquan [1988]).

Proposition 2.6 An initial valuation f such that 0 and 1 are not in D_f is n-stable if an only if

$$x_1\wedge\ldots\wedge x_n\le z \;\Rightarrow\; f(x_1)\wedge\ldots\wedge f(x_n)\le f(z)$$

for every $x_1,\ldots x_n\in D_f$ and $z\in D_f$. Likewise, f is p-stable if and only if

$$x_1\vee\ldots\vee x_n\ge z \;\Rightarrow\; f(x_1)\vee\ldots\vee f(x_n)\ge f(z).$$

for every $x_1,\ldots x_n\in D_f$ and $z\in D_f$.

A simple consequence of Proposition 2.6 is that if D_f is a chain, then f is n-stable (p-stable) if and only if it is increasing.

3. The lattices of the necessities.

Proposition 3.1 in Biacino and Gerla [1990]b shows that N(B) is a complete lattice and that the meets in N(B) coincide with the meets in F(B). The maximum of N(B) is the completely inconsistent necessity, the minimum the characteristic function of the filter {1}. In N(B) there is no atom, indeed if $n\in N(B)$ is not the minimum and we set $n'(x)=n(x)/2$ if $x\ne1$ and $n'(1)=1$, then $n'<n$ and n' is different from the minimum. Likewise one proves that in N(B) there is no co-atom, i.e. no maximal element.

The lattice N(B) is not a sublattice of F(B) since the join of a family $(n_i)_{i\in I}$ of necessities in N(B) is equal to $(\bigvee n_i)^-$, in general. If the condition

$$\forall x\,\forall y\,\forall i\,\forall j\,\exists h\;\; n_i(x)\wedge n_j(y)\le n_h(x)\wedge n_h(y) \qquad (3.1)$$

is satisfied, then it is easily proven that the joins in F(B) coincide with the joins in N(B). In particular this happens for directed families of necessities.

Let n_1 and n_2 be two necessities, we call *sum* n_1+n_2 of n_1 and n_2 its join $(n_1 \vee n_2)^{-}$ in N(B).

Likewise, the class P(B) of the possibilities is a complete lattice and that the joins in P(B) coincide with the joins in F(B). The meet in P(B) of a family $\langle p_i \rangle$ of possibilities is the possibility $(\bigwedge p_i)^{\circ}$ and the property

(3.2) $\forall x \, \forall y \, \forall i \, \forall j \, \exists h \;\; n_i(x) \vee n_j(y) \geq n_h(x) \vee n_h(y)$

assures the coincidence of this possibility with $\bigwedge p_i$. If p_1 and p_2 are two possibilities, the *product* $p_1 \cdot p_2$ is their meet in P(B). The function $\sim : N(B) \to P(B)$ is a dual isomorphism between the lattices N(B) and P(B).

Proposition 3.1 If n_1 and n_2 are two necessities and p_1 and p_2 are two possibilities then, for every $z \in B$

(3.3) $(n_1+n_2)(z) = \bigvee \{n_1(x) \wedge n_2(y) / x \wedge y \leq z\}$; $(p_1 \cdot p_2)(z) = \bigwedge \{p_1(x) \vee p_2(x) / x \vee y \geq z\}$

(3.4) $\sim(n_1+n_2) = (\sim n_1) \cdot (\sim n_2)$; $\sim(p_1 \cdot p_2) = (\sim p_1) + (\sim p_2)$.

(3.5) $C_r(n_1+n_2) = \bigvee \{n_1(x) \wedge n_2(-x) / x \in B\}$; $C_s(p_1 \cdot p_2) = \bigwedge \{n_1(x) \vee n_2(-x) / x \in B\}$.

(3.6) $n^a_\alpha + n^b_\alpha = n^{a \wedge b}_\alpha$; $p^a_\alpha \cdot p^b_\alpha = p^{a \vee b}_\alpha$.

We point out that by Proposition 3.1 we have that, if f_1 and f_2 are initial valuations,

(3.7) $(f_1 \vee f_2)^{-} = \bar{f}_1 + \bar{f}_2$ and $(f_1 \wedge f_2)^{-} = \bar{f}_1 \cdot \bar{f}_2$.

As a consequence, if $D_f = \{e_1, \ldots, e_m\}$ and $f(e_i) = \alpha_i$, then

(3.8) $\bar{f} = n^{e_1}_{\alpha_1} + \ldots + n^{e_m}_{\alpha_m}$

In particular, if $\alpha_1 = \ldots = \alpha_m = \alpha$, and $e = e_1 \wedge \ldots \wedge e_m$ then

(3.9) $\bar{f} = n^e_\alpha$

4. Conditional necessities and possibilities.

Let n be a necessity, $a \in B$ an event and let n^a be the necessity generated by the event a, i.e. the characteristic function of the filter generated by a.

The sum $n+n^a$ is called *the conditional necessity given the event a* and we set

(4.1) $$n(x/a)=(n+n^a)(x).$$

In particular, if n is the necessity generated by an event e, $n^e(\cdot/a)=n^e+n^a$ and, by (3.6), $n^e(\cdot/a)=n^{e\wedge a}$.

Proposition 4.1 Given a necessity n and an event a,

i) $n(x/a) = n(a\to x)$; $n(a/a)=1$, $n(x/1) = n(x)$; $C_r(n(\cdot/a)) = n(-a)$;

ii) $n(x\wedge y/a)=n(x/a)\wedge n(y/a)$; $n(x/a\vee b)=n(x/a)\wedge n(x/b)$;

iii) $n(a)\wedge n(x/a)=n(x\wedge a)$; $n(x)=n(x/a)\wedge n(x/-a)$;

iv) $n(a)\wedge n(x/a) = n(x)\wedge n(a/x)$.

The first equality in i) shows the link between our definition of conditional necessity and the notion given in D.Dubois and H.Prade [1988]. The difference is that they proposed

$$n(x/a) = \begin{cases} n(a\to x) & \text{if } n(-a)<n(a\to x) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if $n(-a)=0$, that is, by the last equality in i), if $n(\cdot/a)$ is completely consistent, the two notions are equivalent.

The equality iv) is analogous to Bayes formula but unfortunately it is not possible to obtain $n(a/x)$ as a function of $n(x/a)$, $n(x)$ and $n(a)$ as in the probabilistic case. Indeed, assume that h is such a function, i.e. $n(x/a)=h(n(a),n(x),n(a/x))$ for every pair of events x, a. In particular, if n is the necessity generated by an event $e\neq 0$, then

$$n^{e\wedge a}(x) = h(n^e(a),n^e(x),n^{e\wedge x}(a)).$$

Now, if e, a and x satisfies $e\wedge x\nleq a$, $a\leq x$, $e\nleq x$, then, since $e\nleq a$ and $e\wedge a\leq x$, we have $h(0,0,0) = 1$. On the other hand, if $e\wedge a\nleq x$ and $e\wedge x\nleq a$, then, since $e\nleq a$, $e\nleq x$, we have $h(0,0,0)=0$, an absurdity.

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