SOME RESULTS ON WEAK t-NORMS*

János C. FODOR

Computer Center, Eötvös Loránd University
H-1502 Budapest 112, P.O. Box 157, Hungary

Abstract The notion of weak t-norms - as a model for intersection of fuzzy sets - was introduced in [1]. In this paper we present some results on the representation of an important class of weak t-norms and some related problems such as the properties of negations based on weak t-norms and comparison of weak t-norms via their generator functions.

1. Introduction

First we recall the notion of a weak t-norm from [1]. Let $I = [0,1]$ and $I_0 = (0,1)$.

Definition 2.1. A function $w: I\times I \to I$ will be called *weak t-norm* if it has the following properties:

$$w(1,a) = a, \quad w(a,1) \leqq a \qquad \forall\, a \in I, \tag{1.1}$$

$$w(a,b) \leqq w(c,d) \quad \text{when} \quad a \leqq c,\ b \leqq d. \tag{1.2}$$

If w is a weak t-norm then its *right pseudocomplement* is defined by

$$w^{\rightarrow}(a,b) = \sup \{ x \,;\, w(a,x) \leqq b \}. \tag{1.3}$$

Denote $\mathcal{W}$ the class of weak t-norms with property

$$w(a,x) \text{ is left-continuous in } x \text{ on } I \text{ for every } a \in I. \tag{1.4}$$

We proved in [1] the following result concerning $w^{\rightarrow}$.

*This research has been partially supported by OTKA-27-5-606 and FKP-400-0113.

Proposition 1.1. If $w \in \mathcal{W}$ then we have

(1) $w^{\rightarrow}(a,x)$ is right-continuous with respect to x on I for every $a \in I$.

(2) If $a \leqq c$ then $w^{\rightarrow}(a,b) \geqq w^{\rightarrow}(c,b)$.

(3) If $b \leqq d$ then $w^{\rightarrow}(a,b) \leqq w^{\rightarrow}(a,d)$.

(4) If $a \leqq b$ then $w^{\rightarrow}(a,b) = 1$.

(5) $w^{\rightarrow}(1,b) = b \quad \forall\, b \in I$.

(6) $c \leqq w^{\rightarrow}(a,b)$ if and only if $w(a,c) \leqq b$. □

Properties (1) - (5) are accepted for a *fuzzy implication function* in the literature, see e.g. Trillas and Valverde [5]. So $w^{\rightarrow}$ seems to be a good model for a fuzzy implication. On the other hand, the class $\mathcal{W}$ is fairly broad not only for the theory but also for the applications, see examples in [1]. However, the Exchange Principle , i.e.,

$$w^{\rightarrow}(a,w^{\rightarrow}(b,c)) = w^{\rightarrow}(b,w^{\rightarrow}(a,c)), \tag{1.5}$$

does not hold automatically when $w \in \mathcal{W}$.

We proved in [2] the next result on the Exchange Principle.

Proposition 1.2. Assume that $w \in \mathcal{W}$. $w^{\rightarrow}$ fulfils (1.5) if and only if

$$w(a,w(b,c)) = w(b,w(a,c)). \ \square$$

In the next section we investigate weak t-norms having property (1.5).

2. Representation of a class of weak t-norms

Assume that $w \in \mathcal{W}$ is such that it has the following properties as well:

i) $\psi(a) = w(a,1)$ is continuous, strictly increasing function

ii) $w(a,w(b,c)) = w(b,w(a,c))$

iii) $w(a,\psi(a)) < \psi(a)$ for $a \in I_0$.

Denote this subclass of $\mathcal{W}$ by $\mathcal{W}_A$. We can present a representation theorem for members of $\mathcal{W}_A$ as follows.

Theorem 2.1. (*Representation theorem*)
$w \in \mathcal{W}_A$ if and only if there exist functions $f,g: I \to \mathbf{R}_+$ with properties

(i) f,g are strictly decreasing, continuous functions

(ii) $g(x) \geqq f(x) \quad \forall\, x \in I$

(iii) $f(1) = g(1) = 0$

such that

$$w(a,b) = f^{(-1)}(\ g(a) + f(b)\ ), \qquad (2.1)$$

where $f^{(-1)}$ denotes the pseudoinverse of f . □

This theorem can be seen as a generalization of the representation theorem for t-norms, see Ling [3] or Schweizer and Sklar [4].

We will call the ordered pair (f,g) additive generators of w if w has the form (2.1) with f and g.

Theorem 2.2. Assume that for a $w \in \mathcal{W}_A$ we have

$$w(a,b) = f_1^{(-1)}(\ g_1(a) + f_1(b)\ ) \ = \ f_2^{(-1)}(\ g_2(a) + f_2(b)\ ).$$

Then there exists an $\alpha > 0$ such that $f_2 = \alpha f_1$ and $g_2 = \alpha g_1$.□

We say that a weak t-norm w *has zero divisors* if there exist $a,b > 0$ such that $w(a,b) = 0$. A weak t-norm w is *strict* if it is strictly increasing on $I_0 \times I_0$.

Theorem 2.3. Assume that $w \in \mathcal{W}_A$ with additive generators (f,g). Then

a) w has zero divisors if and only if $f(0) < +\infty$,

b) w is strict if and only if $f(0) = \lim_{x \to 0} f(x) = +\infty$. □

3. Negations based on weak t-norms

A function $n: I \to I$ is called *negation* if n is nonincreasing and $n(0) = 1$, $n(1) = 0$. A negation is *strict* if n is continuous and decreasing. A strict negation is *strong* if $n(n(a)) = a$ for every $a \in I$.

As in the case of t-norms, one can define a negation by $\vec{w}(a,0)$.

Theorem 3.1. Suppose that $w \in \mathcal{W}_A$. Then

a) $$\vec{w}(a,0) = \begin{cases} 0 & \text{if } a > 0 \\ 1 & \text{if } a = 1 \end{cases}$$ when w is strict;

b) If w has zero divisors then there exists an $a_0 < 1$ such that $\vec{w}(a,0) = 1$ for $a \in [0,a_0]$ and $\vec{w}(a,0)$ is strictly decreasing on $(a_0,1]$. □

Theorem 3.2. Let $w \in \mathcal{W}_A$ be such that it has zero divisors. Then

a) $\vec{w}(a,0)$ is a strict negation <u>iff</u> $g(0) = f(0) < +\infty$;

b) $\vec{w}(a,0)$ is a strong negation <u>iff</u> $g(x) = f(x)$ and $f(0) < +\infty$.□

This means that $\vec{w}(a,0)$ is a strong negation if and only if w is an Archimedian t-norm with zero divisors.

4. Comparison of weak t-norms

It is well-known from the theory of t-norms that one can express the relation $T_1(a,b) \leqq T_2(a,b)$ via the generator functions of T_1 and T_2. This is also the case in connection with weak t-norms.

Theorem 4.1. Assume that w_1, $w_2 \in \mathcal{W}_A$. Then $w_1 \leqq w_2$ if and only if

$$f_1 \circ f_2^{(-1)}(u + v) \leqq g_1 \circ g_2^{(-1)}(u) + f_1 \circ f_2^{(-1)}(v)$$

for every $u,v \in I$, where (f_1,g_1) and (f_2,g_2) are the generator functions of w_1 and w_2, respectively. □

Corollary. If $f_1 \circ f_2^{(-1)}$ is subadditive then $w_1 \leqq w_2$. □

REFERENCES

[1] J.C. Fodor, Strict preference relations based on weak t-norms, *Fuzzy Sets and Systems*, to appear

[2] J.C. Fodor, On fuzzy implication operators, *Fuzzy Sets and Systems*, submitted

[3] C.H. Ling, Representation of associative functions, *Publ. Math. Debrecen* 12 (1965) 189-212.

[4] B. Schweizer and A. Sklar, Probabilistic Metric Spaces (North-Holland, Amsterdam, 1983).

[5] E. Trillas and L. Valverde, On implication and indistinguishability in the setting of fuzzy logic, in: J.Kacprzyk and R.R. Yager, Eds., Management Decision Support Systems Using Fuzzy Sets and Possibility Theory (Verlag TÜV Rheinland, Köln, 1985) 198-212.