THE NUMERICAL METHOD OF BOOLEAN OPERATIONS OF PLANAR REGIONS

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ABSTRACT

A Boolean Function of planar regions, Y=f(A,B,C,...) may be some not overlapping regions. Using several useful theorems provided in this paper and the laws of Boolean operations, we can tell the areas and boundaries of these planar regions according to the boundaries of A,B,C,...The numerical methodnot only makes it possible that the logical operation of line segments and that of planar regions are done in the computer, but also possesses meaning to the fuzzy mathematics.

The Boolean operations can be illustrated by Venn diagrams on a plane. Suppose there are two planay regions, A and B. Using eyes and mind, we can tell the boundaries even areas of regions represented by Y=f(A,B,C,...), the Boolean function of A, B, C ... However, we didn't have an universal method for the computer to do this work until now. In this paper, some useful formulae are provided so that the boundaries and areas of Y=f(A,B,C ...) can be calculated out from the boundies of A B C ...; the Boolean operations of planar regions may become the operation of maximum, minimum addition and subtraction of the boundaries of planar regions.

In order to resolve the problem of the Boolean operations of planar regions, we first resolve the problem of the Boolean operation of line segments.

1. The Boolean operation of line Segments

Let a line I represent a longitudinal interval $[^1_0]$, i.e. $I=[^1_0]$; sub-lines (first suppose all of their lower ends are 0) on I are $A=[^a_0]$, $B=[^b_0]$, $C=[^c_0]$, ...; the Boolean function of A,B,C ... is

$$Y=f(A,B,C\ \ldots).$$

Y may be some sub-lines that do not overlap each other. How to know the upper and lower ends of these sub-lines and $|Y|$, the measurement of Y? this is to be done as following.

Suppose that f(A,B,C ...) contains three kinds of operations , $\vee$, $\wedge$ and $^{-}$, and that the $\wedge$ can be omitted. In order to have $|Y|$ become the function of a,b,c ... that are upper ends of A,B,C ... respectively, let $|\ |$, A,B,C ... in $|f(A,B,C\ \ldots)|$ be substituted by $[\ ]$, a,b,c ... respectively, i.e.

$$|f(A,B,C\ \ldots)|=[f(a,b,c\ \ldots)].$$

According to the above definition, f(a,b,c ...) follows the laws of Boolean operations as well as f(A,B,C ...). Actually, the $[\ ]$ is the different form of $|\ |$. cóncerning $[\ ]$, there are several useful theorems.

Theorem 1. If there are two variables in f(...), we have

$$[ab]=\min(a,b);$$
$$[a\vee b]=\max(a,b);$$
$$[a\bar{b}]=\max\ (0,a-b).$$

The above formulae are obvious. From them, we can deduce

$$[a]=a;$$
$$[\bar{a}]=1-a;$$
$$[a\bar{a}]=0.$$

Theorem 2. If $f(a,b,c\ \ldots)=Y_1\vee Y_2$, Y_1 and Y_2 are subfunctions and $[Y_1Y_2]=0$, then $[f(a,b,c\ \ldots)]=[Y_1]+[Y_2]$.

for example

$$[a\vee\bar{a}]=[a]+[\bar{a}]=1.$$

Proof : since $[Y\ Y]=0$, line segment Y Y=0. Therefore

$$[f(a,b,c\ \ldots)]=|f(A,B,C\ \ldots)|$$
$$=|Y|+|Y|$$
$$=[Y]+[Y].$$

Theorem 3. If w in f(...,w,...) is a sub-function that does not contain the negative operation^{-}, then

$$[f(\ldots,w,\ldots)]=[f(\ldots,\ w\ ,\ldots)].$$

For example

$$[a\bar{b}\bar{c}]=[a\overline{b\vee c}]=[a\overline{[b\vee c]}]=\max(0,a-\max(b,c)).$$

Proof : We only need to prove [w]=w. since w does not contain, W must be a line the lower end of which is 0; |W| must be equal to w that is the upper end of W. And account of [w]=|W|, we have [w]=w.

Depending upon the above theorems and the laws of Boolean operations, it is not difficult to change any complex [f(a b c ...)] into a function that only contains the operations, maximum, minimum, addition and subtraction of a, b, c, ... First, we may change [f (a b c...)] into the sum of several parts, any one of which contains no more than one^{-}. (This is possible after all because if there are n variables in f(...), we are certainly able to change f(...) into the sum of some of 2^n miniterms, and then use Demogon's law to make every miniterm contain no more than one^{-}.) Then, we may use the theorem 2, 3 and 1 in proper order to obtain the final result. In meanwhile, the upper and lower ends of the sub-lines represented by Y that do not overlap each other are given naturally.

Sample 1 . calculate the value of line function Y=ABCD.

Solution:

$$
\begin{aligned}
[Y] &= [a\overline{\bar{b}c\bar{d}}] \\
&= [a\bar{b}(\bar{c}\vee\bar{d})] \\
&= [a\bar{b}\bar{c}\bar{d}]+[a\bar{b}d] \\
&= [a\overline{[b\vee c\vee d]}]+[[ad]\bar{b}] \\
&= \max(0,a-\max(b\ c\ d))+\max(0,\min(a,d)-b).
\end{aligned}
$$

From the above result we may see Y represent the two sub-lines that do not overlap as following

$$\begin{bmatrix} a \\ \\ \max(b,c,d) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \min(a,d) \\ \\ b \end{bmatrix}$$

So far, there is no problem for the resolution of the line-function in which the lower ends of sub-lines are all 0. If the lower end of a sub-line in f(A B C ...) arenot 0, this sub-line can be regarded as the difference of two sub-lines whose lower ends are both 0. For example,

$A=\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ ($a_2 > a_1$ is allowed. If $a_2 > a_1$, then $|A|=0$). we may suppose

$$A_1=\begin{bmatrix} a_1 \\ 0 \end{bmatrix},\quad A_2=\begin{bmatrix} a_2 \\ 0 \end{bmatrix}$$

Hence

$$A=A_1\bar{A}_2$$

and

$$\begin{aligned} |Y| &= |f(A,B,C\ \ldots)| \\ &= |f(A_1,\bar{A}_2,B,C\ \ldots)| \\ &= [f(a\ a.b.c\ \ldots)] \end{aligned}$$

Up to now, there is no problem, too, for the resolution of line functions in which the lower ends of sub-lines are 0 or not.

2. The Boolean Operation of Planar Regions

If the $a_1, a_2, b_1, b_2, c_1, c_2 \ldots$ are all the single value functions of $x \in X$, and I^*X appears a plane, the Boolean Operation of line segments will become that of planar regions.

Sample 2. Ais a planar region whose upper and lower boundaries are $a_1(x)$ and $a_2(x)$ respectively; B is a planar region whose upper and lower boundaries are $b_1(x)$ and $b_2(x)$ respectively. (If the boundary function of a region, say $a_1(x)$ and $a_2(x)$, has no definition as $x=x_0 \in X$, then suppose $a_1(x\ )=a_2(x_0)=0$.) Give out the resolution of $Y=A\bar{B}$. (see fig.1a.)

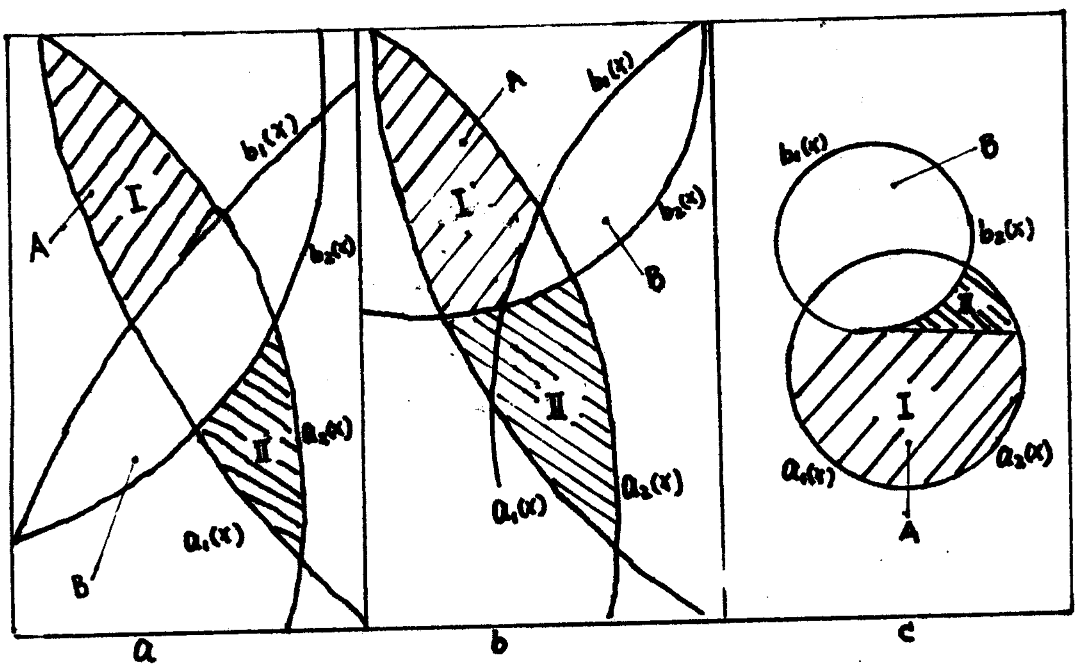

Fig.1 The illustration of the resolution of Boolean function Y=AB

Solution :

$$[Y]=\int[a_1\bar{a}_2\overline{b_1\bar{b}_2}]\,dx=\int(\max_x(0,a_1-\max(a_2,b_1,b_2))+\max(0,\ \min(a_1,b_2)-a_2))\,dx$$

It can been seen that Y represents two sub-regions, I and II, that do not overlap as following

$$\text{I:}\ \begin{bmatrix} a_1 \\ \\ \max(a_1, b_1, b_2) \end{bmatrix} \quad \text{and} \quad \text{II:}\ \begin{bmatrix} \min(a_1, b_2) \\ \\ a_2 \end{bmatrix}$$

It may be verified that the above resolution are always correct in different cases. (see fig.1 b and c.)

3. Summary

The switching algebra may be considered the algebra of points. The algebra of line-segments and the algebra of planar regions should be the extension of the point algebra; they are more general algebras. Though they are the particular samples of Boolean algebra, the above numerical method can be used to define anew algebra that can be called the fuzzy switching quasi-Boolean algebra, which is similar to Zedah algebra but more compatible to Boolean algebra.

The numerical method in this paper seems to have his applications to computer graphics and image processing. Actually, this method was reached with a new mathematical model of color vision being set up. This model is more symmetrical and can give us a more instinct understanding about the mechanism of color vision.

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