L-Fuzzy extended and contracted ideals

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In this note a characterization of the L-fuzzy ideal generated by a L-fuzzy subset is given and then by giving the concepts of L-fuzzy contracted and extended ideals, some related theorems are proved.

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1. Introduction

Zadeh in [9] introduced the notion of a fuzzy subset A of a nonempty set X as a function from X to [0,1]. Goguen [1], generalized the fuzzy subsets of X, to L-fuzzy subsets, a function from X to a lattice L. Since then several authors have developed interesting results on fuzzy ideals of a ring. For example see [3,4,5,6,8,10,11]. In [10], we have given a characterization of a L-fuzzy ideal generated by a L-fuzzy point. In this note, for commutative rings having identity, we give a characterization of the L-fuzzy ideal generated by a L-fuzzy subsets. Then we define the concept of L-fuzzy extended and contracted ideals and we prove that there exists a bijection between the set of all L-fuzzy contracted ideals onto the set of all L-fuzzy extended ideals of homomorphism rings. Moreover we prove that every L-fuzzy ideal of the ring of quotient of a given ring is a L-fuzzy extended ideal. Also a necessary and sufficient condition of a L-fuzzy contracted prime ideal is given, and it is shown that there is a bijection between the set of all L-fuzzy prime ideals of the ring of quotients to a class of L-fuzzy prime ideals of the ring.

2. Preliminaries

We fix $L=(L,\leq,\vee,\wedge)$ as a completely distributive lattice, which has least and greatest elements, say 0 and 1, and for simplicity of notation we write "sup" and "inf" for "$\vee$" and "$\wedge$", respectively. If $a,b\in L$, we write $a\geq b$ if $b\leq a$.

For a nonempty set X, let

$$F(X)=\{A \mid A \text{ is a L-fuzzy subset of } X\}.$$

If $A,B\in F(X)$, then $A\leq B$ if and only if $A(x)\leq B(x)$ for all $x\in X$.

From now on we write R and S for rings.

Let $f:R\to S$ be a map, $A\in F(R)$ and $B\in F(S)$. Then we define $f(A)\in F(S)$ and $f^{-1}(B)\in F(R)$, by

$$f(A)(y)=\begin{cases} \sup\limits_{x\in f^{-1}(y)} A(x) & \text{if} \quad f^{-1}(y)\neq\emptyset \\ 0 & \text{if} \quad f^{-1}(y)=\emptyset, \end{cases}$$

$$f^{-1}(B)(x)=B(f(x)),$$

respectively.

Definition 2.1. Let $A\in F(R)$. Then A is called a L-fuzzy left (right) ideal of R if and only if for all $x,y\in R$.

(i) A is a L-fuzzy subgroup of $(R,+)$; i.e.,

$$A(x-y)\geq \inf\ (A(x),A(y))$$

(ii) $A(xy)\geq A(y)$ $\quad (A(xy)\geq A(x))$.

Note that (i) above implies that $A(x)=A(-x)$. A is called a L-fuzzy ideal of R if and only if it is both L-fuzzy left and L-fuzzy right ideal of R.

We let I(R) be the set of all L-fuzzy ideals of R.

Theorem 2.2 [4, Theorem 4.1]. Let $A \in F(R)$. The L-fuzzy subset $C(x) = \inf_{A \subseteq B \in I(R)} B(x)$ is the smallest L-fuzzy ideal of R containing A, i.e., $A \subseteq C$ and for any $C' \in I(R)$ such that $A \subseteq C'$, then $C \subseteq C'$.

Definition 2.3. The L-fuzzy ideal C in Theorem 2.2 is called the L-fuzzy ideal generated by A and is denoted by $\langle A \rangle$.

Lemma 2.4. Let $f: R \to S$ be a homomorphism. Then

(i) $f(A_1) \subseteq f(A_2)$ if $A_1, A_2 \in F(R)$ and $A_1 \subseteq A_2$

(ii) $f^{-1}(B) \in I(R)$ if $B \in I(S)$

(iii) $f^{-1}(B_1) \subseteq f^{-1}(B_2)$ if $B_1, B_2 \in F(S)$ and $B_1 \subseteq B_2$

(iv) $A \subseteq f^{-1}(f(A))$ if $A \in I(R)$

(v) $f(f^{-1}(B)) \subseteq B$ if $B \in I(S)$.

Theorem 2.5. [8, Theorem 2.1]. (a). Let **p** be a L-fuzzy prime ideal of R and α a prime element in L. Let $P \in F(R)$, defined by $P(x) = \begin{cases} 1 & \text{if } x \in \mathbf{p} \\ \alpha & \text{otherwise} \end{cases}$, then P is a L-fuzzy prime ideal.

(b) Conversely any L-fuzzy prime ideal can be obtained as above.

3. L-fuzzy extended and contracted ideals

From now on all rings are commutative with identity.

Theorem 3.1. Let $A\in F(R)$. Then $\langle A\rangle$ is equal to B, where B is defined as follows

$$B(w) = \sup_{w=\sum_{i=1}^{n} x_i a_i \;;\; \text{for some } x_i, a_i\in R,\ n\in \mathbb{N}} \inf(A(a_1),\dots,A(a_n)).$$

Definition 3.2. Let $f:R\longrightarrow S$ be a homomorphism, and $I\in I(R)$, $J\in I(S)$. Then $I^{e}=\langle f(I)\rangle\in I(S)$ and $J^{c}=f^{-1}(J)\in I(R)$, are called the L-fuzzy extension of I and the L-fuzzy contraction of J under f, respectively.

Definition 3.3. $J\in I(S)$ is called a L-fuzzy extended ideal if $J=I^{e}$ for some $I\in I(R)$ and $I\in I(R)$ is called a L-fuzzy contracted ideal if $I=J^{c}$ for some $J\in I(S)$.

Lemma 3.4. (i) If $I,I'\in F(R)$ are such that $I\subseteq I'$, then $\langle I\rangle\subseteq\langle I'\rangle$

(ii) If $I\in I(R)$, then $I\subseteq I^{ec}=(I^{e})^{c}$

(iii) If $J\in I(S)$, then $J^{ce}=(J^{c})^{e}\subseteq J$.

Lemma 3.5. For $I\in I(R)$ and $J\in I(S)$,

(i) $I^{ece}=I^{e}$,

(ii) $J^{cec}=J^{c}$.

Theorem 3.6. Let $f:R\longrightarrow S$ be a homomorphism. Then the mapping $I\longmapsto I^{e}$ is a bijection between the set, $\mathcal{C}$, of all L-fuzzy contracted ideals in R onto the set, $\mathcal{E}$, of all L-fuzzy extended ideals in S.

In the following theorem by R_S we mean the ring of quotient of R, where S is a multiplicative set as in [7, page 41].

Theorem 3.7. Let S be a multiplicative set and $f:R\rightarrow R_S$ be defined by $f(a)=(a/1)$. Then every $J\in I(R_S)$ is a L-fuzzy extended ideal.

Theorem 3.8. Let $f:R\rightarrow S$ be a homomorphism of rings and P a L-fuzzy prime ideal of R. Then P is the contraction of a L-fuzzy prime ideal of S if and only if $P=P^{ec}$.

Theorem 3.9. Let $f:R\rightarrow S$ be a homomorphism of rings. If S is a faithfully falt R-module; then every L-fuzzy prime ideal of R is the contraction of a L-fuzzy prime ideal of S.

Theorem 3.10. The L-fuzzy prime ideals of R_S are in 1-1 correspondence with the L-fuzzy prime ideals of R, which attains a prime element of L on S.

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