Fuzzy HX Group

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Abstract

In the paper[1] the upgrade of algebraic structure has been considered, in which the concept of HX group has been raised, With the development of fuzzy set theory, all kinds of the structure are upgraded not only from their universes to their power sets but also from their universes to their power fuzzy sets. In this paper, the concept of fuzzy HX group will be first raised and the structure and homomorphisms of fuzzy HX group will be studied.

Key words: fuzzy HX group, uniform fuzzy HX group, regular fuzzy HX group.

1. Fuzzy HX Group

We always assume that X is a group in the paper. The set of all fuzzy sets of X is called the power fuzzy set, denoted by $\mathcal{F}(X)$.

By using multivariate extension principle[2], the operation of the group X can be extended to $\mathcal{F}(X)$. For $\forall \underset{\sim}{A}, \underset{\sim}{B} \in \mathcal{F}(X)$

$$\underset{\sim}{A}\underset{\sim}{B} = \bigcup_{\lambda \in (0,1]} \lambda (A_\lambda B_\lambda) \tag{1.1}$$

where A_λ, B_λ are the λ-cut sets of $\underset{\sim}{A}, \underset{\sim}{B}$, and $A_\lambda B_\lambda = \{ab \mid a \in A_\lambda,\ b \in B_\lambda\}$.
We appoint that $\phi A = A\phi = \phi$.

Proposition 1.1. Let $\underset{\sim}{A}, \underset{\sim}{B} \in \mathcal{F}(X)$, then

$$\underset{\sim}{A}\underset{\sim}{B}(x) = \bigvee_{\substack{yz=x \\ y,z\in X}} (\underset{\sim}{A}(y) \wedge \underset{\sim}{B}(z)) = \bigvee_{y\in X} (\underset{\sim}{A}(y) \wedge \underset{\sim}{B}(y^{-1}x)), \quad \text{for } \forall x \in X.$$

Proposition 1.2. Let $\underset{\sim}{A}, \underset{\sim}{B} \in \mathcal{F}(X)$, we have

$$\underset{\sim}{A}\underset{\sim}{B} = \bigcup_{\lambda\in[0,1]} \lambda(A_{\underset{\cdot}{\lambda}} B_{\underset{\cdot}{\lambda}}) \tag{1.2}$$

where $A_{\underset{\cdot}{\lambda}}, B_{\underset{\cdot}{\lambda}}$ are the strong λ-cut sets of $\underset{\sim}{A}, \underset{\sim}{B}$.

For more general case, we have

Proposition 1.3. If $\underset{\sim}{A} = \bigcup_{\lambda\in[0,1]} \lambda H_A(\lambda)$, $\underset{\sim}{B} = \bigcup_{\lambda\in[0,1]} \lambda H_B(\lambda)$, where H_A, as well as H_B, is a nest of sets on X [2], $A_{\underset{\cdot}{\lambda}} \subseteq H_A(\lambda) \subseteq A_\lambda$, $B_{\underset{\cdot}{\lambda}} \subseteq H_B(\lambda) \subseteq B_\lambda$, for $\forall \lambda \in [0,1]$, then

$$\underset{\sim}{A}\underset{\sim}{B} = \bigcup_{\lambda\in[0,1]} \lambda(H_A(\lambda) H_B(\lambda)) \tag{1.3}$$

and $(\underset{\sim}{A}\underset{\sim}{B})_\lambda = \bigcap_{\alpha<\lambda} H_A(\alpha) H_B(\alpha)$, $(\underset{\sim}{A}\underset{\sim}{B})_{\underset{\cdot}{\lambda}} = \bigcup_{\alpha>\lambda} H_A(\alpha) H_B(\alpha) = A_{\underset{\cdot}{\lambda}} B_{\underset{\cdot}{\lambda}}$.

Notice that $(\underset{\sim}{A}\underset{\sim}{B})_\lambda \neq A_\lambda B_\lambda$.

Proposition 1.4. For $\forall \underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C} \in \mathcal{F}(X)$, we have

$$(\underset{\sim}{A}\underset{\sim}{B})\underset{\sim}{C} = \underset{\sim}{A}(\underset{\sim}{B}\underset{\sim}{C}).$$

According to above discussion, we know that $\mathcal{F}(X)$ is a semigroup with unit element $\chi_{\{e\}}$ for the operation (1.1), where e is the unit element of X, and

$$\chi_{\{e\}}(x) = \begin{cases} 1, & x = e \\ 0, & x \neq e \end{cases}, \quad \text{for } \forall x \in X.$$

Definition 1.1. Let $\mathcal{A} \subseteq \mathcal{F}(X)$. $\mathcal{A}$ is called a fuzzy HX group on X, if $\mathcal{A}$ forms a group for the operation (1.1), which its unit element is denoted by $\underset{\sim}{E}$.

We appoint that empty ϕ is a fuzzy HX group. A fuzzy quotient group on X must be a fuzzy HX group on X and its unit element be a fuzzy normal subgroup of X. A HX group on X must be a fuzzy HX group on X.

Definition 1.2. If $\underset{\sim}{A} \in \mathcal{F}(X)$, then

1). $\underset{\sim}{A}$ is called a fuzzy submonoid on X if for $\forall \lambda \in [0,1]$, $A_{\underset{\cdot}{\lambda}}$ is a submonoid of X.

2). $\underset{\sim}{A}$ is called a fuzzy subsemigroup on X if for $\forall \lambda \in [0,1]$, $A_{\underset{\cdot}{\lambda}}$ is a subsemigroup of X.

Where we appoint that ϕ is a submonoid of X, and is also a subsemigroup of X.

Let $\underset{\sim}{A}\in\mathcal{F}(X)$, we can prove that

1). $\underset{\sim}{A}$ is a fuzzy submonoid on X iff for $\forall\lambda\in[0,1]$, A_λ is a submonoid of X.

2). A is a fuzzy subsemigroup on X iff for $\forall\lambda\in[0,1]$, A_λ is a subsemigroup of X.

Theorem 1.1. Let $\mathcal{A}$ be a fuzzy HX group on X and $\underset{\sim}{E}$ be its unit element, then $\underset{\sim}{E}$ is a fuzzy subsemigroup on X. Conversely, if $\underset{\sim}{A}\in\mathcal{A}$ is a fuzzy submonoid on X, then $\underset{\sim}{A}=\underset{\sim}{E}$.

Proof. $\underset{\sim}{E}$ is the unit element of $\mathcal{A} \Rightarrow \underset{\sim}{E}\underset{\sim}{E}=\underset{\sim}{E} \Rightarrow E_\lambda E_\lambda=E_\lambda$, for $\forall\lambda\in[0,1] \Rightarrow E_\lambda$ is a subsemigroup. $\Rightarrow \underset{\sim}{E}$ is a fuzzy subsemigroup on X. Conversely, $\underset{\sim}{A}$ is a fuzzy submonoid on $X \Rightarrow$ for $\forall\lambda\in[0,1]$, A_λ is a submonoid of $X \Rightarrow A_\lambda A_\lambda=A_\lambda$, for $\forall\lambda\in[0,1]$.

$\Rightarrow \underset{\sim}{A}\underset{\sim}{A}=\bigcup_{\lambda\in[0,1]}\lambda(A_\lambda A_\lambda)=\bigcup_{\lambda\in[0,1]}\lambda A_\lambda=\underset{\sim}{A} \Rightarrow \underset{\sim}{A}=\underset{\sim}{E}$.

Theorem 1.2. Let $\mathcal{A}$ be a fuzzy HX group on X and $\underset{\sim}{E}$ be its unit element, then for $\forall\underset{\sim}{A}\in\mathcal{A}$, $\underset{\sim}{A}$ is as high as $\underset{\sim}{E}$, i.e.

$$\text{high}(\underset{\sim}{A})=\text{high}(\underset{\sim}{E})$$

where $\text{high}(\underset{\sim}{E})\triangleq\bigvee_{x\in X}\underset{\sim}{E}(x)$.

Proof. Because $\underset{\sim}{E}$ is the unit element of $\mathcal{A}$, for $\forall\underset{\sim}{A}\in\mathcal{A}$, $\underset{\sim}{A}\underset{\sim}{E}=\underset{\sim}{A} \Rightarrow A_\lambda E_\lambda=A_\lambda$, for $\forall\lambda\in[0,1] \Rightarrow$ (if $E_\lambda=\phi$, then $A_\lambda=\phi$).

On the other hand, $\underset{\sim}{A}\underset{\sim}{A}^{-1}=\underset{\sim}{E} \Rightarrow A_\lambda(\underset{\sim}{A}^{-1})_\lambda=E_\lambda \Rightarrow$ (if $A_\lambda=\phi$, then $E_\lambda=\phi$).

So $\text{high}(\underset{\sim}{A})=\text{high}(\underset{\sim}{E})$.

Theorem 1.2 shows that the height of all elements in a fuzzy HX group are equal. So, we can define $\text{high}(\underset{\sim}{E})$ the height of fuzzy HX group, write $h(\mathcal{A})$. It is a very important numeral characteristic.

Definition 1.3. Let $\mathcal{A}$ be a fuzzy HX group on X, and $h(\mathcal{A})$ be the height of $\mathcal{A}$. For $\underset{\sim}{A}\in\mathcal{A}$, if there exists $x\in X$ such that $\underset{\sim}{A}(x)=h(\mathcal{A})$, then $\underset{\sim}{A}$ is called a reaching height element.

$\mathcal{A}$ is called a reaching height fuzzy HX group if every element in $\mathcal{A}$ is a reaching height element.

Definition 1.4. Let $\mathcal{A}$ be a fuzzy HX group on X, $\mathcal{A}$ is called conditional strong reaching height, if $\mathcal{A}$ has the properties: for $\forall \underset{\sim}{A}, \underset{\sim}{B} \in \mathcal{A}$, if $\underset{\sim}{A}\underset{\sim}{B}$ reaches the height at x, then there exist $x_1, x_2 \in X$ such that $x = x_1 x_2$, and $\underset{\sim}{A}$ reaches the height at x_1 and $\underset{\sim}{B}$ reaches the height at x_2.

Proposition 1.5. Let $\mathcal{A}$ be a fuzzy HX group. If $\mathcal{A}$ is conditional strong reaching height and $\mathcal{A}$ has a reaching height element, then $\mathcal{A}$ is a reaching height fuzzy HX group.

Proof. Obvious.

Theorem 1.3. Let $\mathcal{A}$ be a conditional strong reaching height fuzzy HX group and $h(\mathcal{A})$ be its height, then

$$\mathcal{A}_{h(\mathcal{A})} \triangleq \{ A_{h(\mathcal{A})} \mid \underset{\sim}{A} \in \mathcal{A} \}$$

is a HX group on X, $E_{h(\mathcal{A})}$ is the unit element of $\mathcal{A}_{h(\mathcal{A})}$, and $A^{-1}_{h(\mathcal{A})} = (\underset{\sim}{A}^{-1})_{h(\mathcal{A})}$.

The proof is easy, so it is omitted.

About general fuzzy HX groups, we have

Theorem 1.4. Let $\mathcal{A}$ be a fuzzy HX group, then for $\forall \lambda \in [0,1]$, $\mathcal{A}_\lambda \triangleq \{A_\lambda \mid \underset{\sim}{A} \in \mathcal{A}\}$ is a HX group on X, and E_λ is the unit element of $\mathcal{A}_\lambda$, $A^{-1}_\lambda = (\underset{\sim}{A}^{-1})_\lambda$.

2. Uniform fuzzy HX group

If the operation of inverse element in X is upgraded to $\mathcal{F}(X)$ by means of extension principle, then we can seek an inverse fuzzy set in $\mathcal{F}(X)$.

Definition 2.1. Let X be a group,

1). For $A \in \mathcal{P}(X)$, $A^{\ominus} \triangleq \{x^{-1} \mid x \in A\}$ is called an inverse set of A.

2). For $\underset{\sim}{A} \in \mathcal{F}(X)$, $\underset{\sim}{A}^{\ominus} \triangleq \bigcup_{\lambda \in [0,1]} \lambda (A_\lambda^{\ominus})$ is called an inverse fuzzy set of $\underset{\sim}{A}$.

By extension priciple, we have

1). $\underset{\sim}{A}^{\ominus} = \bigcup_{\lambda \in [0,1]} \lambda (A_\lambda)^{\ominus}$.

2). $(\underset{\sim}{A}^{\ominus})_\lambda = (A_\lambda)^{\ominus}$.

Generally, in a fuzzy HX group $\mathscr{A}$, the inverse fuzzy set of $\underset{\sim}{A}$ is not uniform with the inverse element of $\underset{\sim}{A}$. For example, let X be the additive group of real numbers, take $\underset{\sim}{E}=(0,+\infty)$, then

$$\mathscr{A} \triangleq \{x+\underset{\sim}{E} \mid x\in X\}$$

is a fuzzy HX group on X, and its unit element is just $\underset{\sim}{E}$. Obviously, $\underset{\sim}{E}^{\ominus}=(-\infty,0)$, but $\underset{\sim}{E}^{-1}=\underset{\sim}{E}$.

In this section we will discuss this kind of fuzzy HX group in which the inverse fuzzy set is uniform with the inverse element.

Definition 2.2. A fuzzy HX group $\mathscr{A}$ is called an uniform fuzzy HX group if for $\forall \underset{\sim}{A}\in\mathscr{A}$, $\underset{\sim}{A}^{\ominus}=\underset{\sim}{A}^{-1}$.

Theorem 2.1. Let $\mathscr{A}$ be a fuzzy HX group, then, $\mathscr{A}$ is uniform iff its unit element $\underset{\sim}{E}$ is a fuzzy subgroup on X.

Proof. 1). Let $\mathscr{A}$ be an uniform fuzzy HX group, then, $\underset{\sim}{E}^{-1}=\underset{\sim}{E}^{\ominus}$ $\Rightarrow (\underset{\sim}{E}^{-1})_\lambda=E_\lambda^{\ominus}$ for $\forall\lambda\in[0,1]$ $\Rightarrow$ For $\forall a,b\in E_\lambda$, $ab^{-1}\in E_\lambda(E_\lambda)^{-1}=E_\lambda(\underset{\sim}{E}^{-1})_\lambda=(\underset{\sim}{E}\underset{\sim}{E}^{-1})_\lambda$ $=E_\lambda \Rightarrow E_\lambda$ is a subgroup of X, for $\forall\lambda\in[0,1]$. $\Rightarrow \underset{\sim}{E}$ is a fuzzy subgroup on X.

2). Let $\underset{\sim}{E}$ be a fuzzy subgroup on X, then, for $\forall\lambda\in[0,1]$, the unit element E_λ of $\mathscr{A}_\lambda$ is a subgroup of X. First of all we prove that for $\forall A_\lambda\in\mathscr{A}_\lambda$, $\forall a\in A_\lambda$, $A_\lambda=aE_\lambda=E_\lambda a$. Clearly $aE_\lambda\subseteq A_\lambda E_\lambda=A_\lambda$. If $aE_\lambda\subsetneq A_\lambda$ then $\exists b\in A_\lambda$, but $b\notin aE_\lambda$. We have $b^{-1}a\notin E$. For $d\in A_\lambda^{-1}$, $(db)^{-1}(da)\in E_\lambda E_\lambda=E_\lambda$ i.e. $b^{-1}a\in E$. This is in contradication with $b^{-1}a\notin E$. So $A_\lambda=aE_\lambda$. Similarly we have $A_\lambda=E_\lambda a$.

Next, we prove $A_\lambda^{-1}=(A_\lambda)^{\ominus}$, for $\forall\lambda\in[0,1]$. For $\forall a\in A_\lambda^{-1}$, noting $A_\lambda^{-1}A_\lambda=E_\lambda$ and $e\in E_\lambda$, then $\exists b\in A_\lambda^{-1}$, $b'\in A_\lambda$, such that $bb'=e \Rightarrow b^{-1}=b'\in A_\lambda$. By above proof we have $A_\lambda^{-1}=bE_\lambda \Rightarrow \exists c\in E_\lambda$ such that $a=bc$. $\Rightarrow a^{-1}=c^{-1}b^{-1}\in E_\lambda b^{-1}=A_\lambda \Rightarrow a\in A_\lambda^{\ominus}$, So $A_\lambda^{-1}\subseteq A_\lambda^{\ominus}$. Conversely, for $a\in A_\lambda^{\ominus} \Rightarrow a^{-1}\in A_\lambda$. Similalarly we can prove $a\in A_\lambda^{-1}$. So $A_\lambda^{\ominus}\subseteq A_\lambda^{-1}$.

Thus $A_\lambda^{-1}=A_\lambda^{\ominus}$, for $\forall\lambda\in[0,1]$.

Therefore $\underset{\sim}{A}^{\ominus}=\bigcup_{\lambda\in[0,1]}\lambda(A_\lambda)^{\ominus}=\bigcup_{\lambda\in[0,1]}\lambda(A_\lambda^{-1})=\bigcup_{\lambda\in[0,1]}\lambda(\underset{\sim}{A}^{-1})_\lambda=\underset{\sim}{A}^{-1}$.

By definition 2.2, $\mathscr{A}$ is an uniform fuzzy HX group on X.

In order to discuss the structure of uniform fuzzy HX groups we will give some new concepts.

Definition 2.3. Let A be a subgroup of X.

1). A subgroup E of X is called a pseudo-normal subgroup for A if for $\forall a\in A$, $aE=Ea$.

2). A fuzzy subgroup $\underset{\sim}{E}$ of X is called a pseudo-normal fuzzy subgroup for A if for $\forall\lambda\in[0,1]$, E_λ is a pseudo- normal subgroup for A.

<u>Proposition 2.1</u>. $\underset{\sim}{E}$ is a pseudo-normal fuzzy subgroup for A iff for $\forall a\in A$, $a\underset{\sim}{E}=\underset{\sim}{E}a$.

<u>Theorem 2.2</u>. 1). If E is a pseudo-normal subgroup for A, then

$$A/E \triangleq \{aE \mid a\in A\}$$

is a HX group on X, and the unit element is just E.

2). If $\underset{\sim}{E}$ is a pseudo-normal fuzzy subgroup for A, then

$$A/\underset{\sim}{E} \triangleq \{a\underset{\sim}{E} \mid a\in A\}$$

is a fuzzy HX group on X, and the unit elemeny is just $\underset{\sim}{E}$.

<u>Definition 2.4</u>. A/E is called a pseudo-quotient group of A for E, and $A/\underset{\sim}{E}$ is called a pseudo-fuzzy quotient group of A for $\underset{\sim}{E}$.

<u>Proposition 2.2</u>. Let A and E be subgroups of X, then the following four conditions are equivalent:

1). E is a pseudo-normal subgroup for A.

2). $aEa^{-1}=E$, for $\forall a\in A$.

3). $aEa^{-1}\leq E$, for $\forall a\in A$.

4). $aha^{-1}\in E$, for $\forall a\in A$, $\forall h\in E$.

The proof is straight.

For reaching height fuzzy HX group $\mathscr{A}$, let $h(\mathscr{A})$ be the height of $\mathscr{A}$, then, for $\forall \underset{\sim}{A}\in\mathscr{A}$, we have $A_{h(\mathscr{A})}\neq\phi$, write $X^*\triangleq\bigcup_{\underset{\sim}{A}\in\mathscr{A}}A_{h(\mathscr{A})}$. Now we show a structure theorem.

<u>Theorem 2.3</u>. Let $\mathscr{A}$ be a uniform fuzzy HX group and its unit element be $\underset{\sim}{E}$. If $\mathscr{A}$ is reaching height, then, X^* is a subgroup of X, $\underset{\sim}{E}$ is a pseudo-normal fuzzy subgroup, and $\mathscr{A}=X^*/\underset{\sim}{E}$.

<u>Proof</u>. 1). For $\forall a,b\in X^*$, $\exists\, \underset{\sim}{A},\underset{\sim}{B}\in\mathscr{A}$, such that $a\in A_{h(\mathscr{A})}$, $b\in B_{h(\mathscr{A})}$ $\Longrightarrow$ For $\forall\lambda<h(\mathscr{A})$, we have $a\in A_\lambda$, $b\in B_\lambda$ $\Longrightarrow$ $ab^{-1}\in A_\lambda(B_\lambda)^{\ominus}=A_\lambda B_\lambda^{-1}=(\underset{\sim}{A}\underset{\sim}{B}^{-1})_\lambda$, where $\underset{\sim}{A}\underset{\sim}{B}^{-1}\in\mathscr{A}$ $\Longrightarrow ab^{-1}\in\bigcap_{\lambda<h(\mathscr{A})}(\underset{\sim}{A}\underset{\sim}{B}^{-1})_\lambda=(\underset{\sim}{A}\underset{\sim}{B}^{-1})_{h(\mathscr{A})}\subseteq X^*$.

So X^* is a subgroup of X.

2). For $\lambda<h(\mathscr{A})$, $h\in E_\lambda$, $a\in X^*$; there exists $\underset{\sim}{A}\in\mathscr{A}$, such that $a\in A_{h(\mathscr{A})}\subseteq A_\lambda$. We have $a^{-1}ha\in A_\lambda^{\ominus}E_\lambda A_\lambda=A_\lambda^{-1}A_\lambda=E_\lambda$ $\Longrightarrow$ E_λ is a pseudo-normal subgroup of X^*. For $\lambda\geq h(\mathscr{A})$, $E_\lambda=\phi$. So $\underset{\sim}{E}$ is a normal fuzzy subgroup on X.

3). For $\forall \underset{\sim}{A} \in \mathscr{A}$, since $\mathscr{A}$ is reaching height, $A_{h(\mathscr{A})} \neq \phi$. Taking $a \in A_{h(\mathscr{A})} \subseteq X^*$, for $\forall \lambda < h(\mathscr{A})$, we have $a \in A_\lambda \Rightarrow aE_\lambda \subseteq A_\lambda E_\lambda = A_\lambda$. Conversely, for $\forall a_1 \in A_\lambda$, $a = ea_1 = aa^{-1}a_1 \in aA_\lambda^{\ominus} A_\lambda = aA_\lambda^{-1} A_\lambda = aE_\lambda$. So $A_\lambda = aE_\lambda$. Thus $\underset{\sim}{A} = \bigsqcup_{\lambda \in (0,1]} \lambda A_\lambda = \bigsqcup_{\lambda \in (0,1]} \lambda (aE_\lambda) = a\underset{\sim}{E} \in X^*/\underset{\sim}{E}$. Conversely, for $\forall a\underset{\sim}{E} \in X^*/\underset{\sim}{E}$, where $a \in X^*$, we have $\underset{\sim}{A} \in \mathscr{A}$, such that $a \in A_{h(\mathscr{A})}$. Similarly, we have $a\underset{\sim}{E} = \underset{\sim}{A} \in \mathscr{A}$.

Therefore $\mathscr{A} = X^*/\underset{\sim}{E}$.

This completes the proof.

Corallary 1. If the step of every element of X is finite, then, a fuzzy HX group $\mathscr{A}$ on X is uniform. If $\mathscr{A}$ is reaching height, then, $\mathscr{A} = X^*/\underset{\sim}{E}$, where $\underset{\sim}{E}$ is the unit element of $\mathscr{A}$.

Corallary 2. If X is a finite group, then, a fuzzy HX group $\mathscr{A}$ on X is reaching height and uniform, therefore $\mathscr{A} = X^*/\underset{\sim}{E}$, where $\underset{\sim}{E}$ is the unit element of $\mathscr{A}$.

3. Regular fuzzy HX group

Definition 3.1. Let $\mathscr{A}$ be a fuzzy HX group on X, $\mathscr{A}$ is called a regular fuzzy HX group if its unit element $\underset{\sim}{E}$ is a fuzzy submonoid on X.

Proposition 3.1. If $\mathscr{A}$ is a regular fuzzy HX group on X, then $\mathscr{A}_\lambda \triangleq \{A_\lambda \mid \underset{\sim}{A} \in \mathscr{A}\}$ is a regular HX group on X.

Definition 3.2. Let A be a subgroup of X.

1). A submonoid E of X is called a pseudo-normal subsemigroup for A if for $\forall a \in A$, $aE = Ea$.

2). A fuzzy submonoid $\underset{\sim}{E}$ of X is called a pseudo-normal fuzzy subsemigroup for A if for $\forall \lambda \in [0,1]$, E_λ is a pseudo-normal subsemigroup for A.

Proposition 3.2. Let A be a subgroup of X and E be a submonoid of X, then, the following four conditions are equivalent:

1). E is a pseudo-normal subsemigroup for A.
2). $aEa^{-1} = E$, for $\forall a \in A$.
3). $aEa^{-1} \subseteq E$, for $\forall a \in A$.
4). $aha^{-1} \in E$, for $\forall a \in A$, $\forall h \in E$.

The proof is straight.

Theorem 3.1. 1). If E is a pseudo-normal subsemigroup for A, then, $A \stackrel{!}{;} E \triangleq \{aE \mid a \in A\}$ is a HX group on X, its element is just E.

2). If $\underset{\sim}{E}$ is a pseudo-normal fuzzy subsemigroup for A, then $A \stackrel{!}{;} \underset{\sim}{E} \triangleq \{ a\underset{\sim}{E} \mid a \in A\}$ is a fuzzy HX group on X, and its unit element is just $\underset{\sim}{E}$.

Definition 3.3. $A \stackrel{!}{;} E$ is called a pseudo quasi-quotient group of A for E, and $A \stackrel{!}{;} \underset{\sim}{E}$ is called a pseudo quasi-fuzzy quotient group of A for $\underset{\sim}{E}$.

Now we discuss the structure of regular fuzzy HX groups.

Let $\mathscr{A}$ be a fuzzy HX group, write

$$\overline{X} \triangleq \bigcup \{ \overline{A}_{h(\mathscr{A})} \mid \underset{\sim}{A} \in \mathscr{A} \}$$

where $h(\mathscr{A})$ is the height of $\mathscr{A}$, $\overline{A}_{h(\mathscr{A})} \triangleq \{ a \mid a \in A_{h(\mathscr{A})},\ a^{-1} \in A^{-1}_{h(\mathscr{A})} \}$

If $\mathscr{A}$ is conditional strong reaching height, then

1). $e \in E_{h(\mathscr{A})} \Rightarrow \overline{A}_{h(\mathscr{A})} \neq \phi$, for $\forall A_{h(\mathscr{A})} \in \mathscr{A}_{h(\mathscr{A})}$; $\exists A_{h(\mathscr{A})} \in \mathscr{A}_{h(\mathscr{A})}$ such that $\overline{A}_{h(\mathscr{A})} \neq \phi \Rightarrow e \in E_{h(\mathscr{A})}$.

2). $\overline{X} \neq \phi \iff e \in E_{h(\mathscr{A})}$.

Theorem 3.2. Let $\mathscr{A}$ be a regular fuzzy HX group on X and $\underset{\sim}{E}$ be its unit element. If $\mathscr{A}$ is conditional strong reaching height, then , $\overline{X}$ is a subgroup of X, $\underset{\sim}{E}$ is a pseudo-normal fuzzy subsemigroup for $\overline{X}$, and $\mathscr{A}$ is a pseudo quasi-fuzzy quotient group of $\overline{X}$ for $\underset{\sim}{E}$, i.e. $\mathscr{A} = \overline{X} \stackrel{!}{;} \underset{\sim}{E}$.

Proof. 1). Since $\mathscr{A}$ is a regular fuzzy HX group on X, $\underset{\sim}{E}$ is a fuzzy submonoid on X. $\Rightarrow E_{h(\mathscr{A})} \neq \phi$, $E_{h(\mathscr{A})}$ is a submonoid of X. Since $\mathscr{A}$ is conditional strong reaching height, by theorem 1.3, we have $\mathscr{A}_{h(\mathscr{A})} = \{ A_{h(\mathscr{A})} \mid \underset{\sim}{A} \in \mathscr{A} \}$ is a regular HX group on X. From [1] , $\overline{X}$ is a subgroup of X.

2). For $\forall \lambda < h(\mathscr{A})$, $h \in E_\lambda$, $a \in \overline{X}$, there exists $\underset{\sim}{A} \in \mathscr{A}$, such that $a \in \overline{A}_{h(\mathscr{A})} \Rightarrow a \in A_{h(\mathscr{A})} \subseteq A_\lambda$, $a^{-1} \in A^{-1}_{h(\mathscr{A})} \subseteq A^{-1}_\lambda \Rightarrow aha^{-1} \in A_\lambda E_\lambda A^{-1}_\lambda = E_\lambda$. By proposition 3.2, E_λ is a pseudo-normal subsemigroup for $\overline{X}$. For $\forall \lambda \geqslant h(\mathscr{A})$, we have $E_\lambda = \phi$. Therefore, $\underset{\sim}{E}$ is a pseudo-normal fuzzy subsemigroup for $\overline{X}$.

3). Since $\mathscr{A}$ is regular, $\underset{\sim}{E}$ reaches the height of $\mathscr{A}$. $\mathscr{A}$ is conditional strong reaching height, so $\mathscr{A}$ is reaching height. $\Rightarrow \overline{A}_{h(\mathscr{A})} \neq \phi$, for $\forall \underset{\sim}{A} \in \mathscr{A}$. Taking $a \in \overline{A}_{h(\mathscr{A})} \subseteq \overline{X}$, we have that $a \in \overline{A}_\lambda$, for $\forall \lambda < h(\mathscr{A})$. Clearly, $aE_\lambda \subseteq A_\lambda E_\lambda = A_\lambda$. In the other respect, $a_1 \in A_\lambda$ implies $a_1 = ea_1 = aa^{-1}a_1 \in aA^{-1}_\lambda A_\lambda = aE_\lambda$. So $A_\lambda \subseteq aE_\lambda$. Thus $A_\lambda = aE_\lambda$.

Therefore $\underset{\sim}{A} = \bigcup_{\lambda\in[0,1]} \lambda A_\lambda = \bigcup_{\lambda\in[0,1]} \lambda(aE_\lambda) = a\underset{\sim}{E} \in \bar{X} \mid \underset{\sim}{E}$.

Conversely, for $\forall a\underset{\sim}{E} \in \bar{X} \mid \underset{\sim}{E}$, where $a\in\bar{X}$, we have that $\exists \underset{\sim}{A} \in \mathcal{A}$ such that $a\in \bar{A}_{h(\mathcal{A})}$. Similarly we have $a\underset{\sim}{E} = \underset{\sim}{A} \in \mathcal{A}$. So $\mathcal{A} = \bar{X} \mid \underset{\sim}{E}$.

4. Homomorphism in fuzzy HX group

Theorem 4.1. Let f be a homomorphism from X to another group Y. If $\mathcal{A}$ is a fuzzy HX group on X, then

$$\mathcal{B} \triangleq f(\mathcal{A}) \triangleq \{f(\underset{\sim}{A}) \mid \underset{\sim}{A} \in \mathcal{A}\}$$

is a fuzzy HX group on Y, $\mathcal{A} \sim \mathcal{B}$, and

1). If $\mathcal{A}$ is uniform then so is $\mathcal{B}$.

2). If $\mathcal{A}$ is regular then so is $\mathcal{B}$.

3). $h(\mathcal{A}) = h(\mathcal{B})$.

4). If $\mathcal{A}$ is reaching height then so is $\mathcal{B}$.

Theorem 4.2. Let f be a surjective homomorphism from X to another group Y. If $\mathcal{B}$ is a fuzzy HX group on Y, then

$$\mathcal{A} \triangleq f^{-1}(\mathcal{B}) \triangleq \{f^{-1}(\underset{\sim}{B}) \mid \underset{\sim}{B} \in \mathcal{B}\}$$

is a fuzzy HX group on X, $\mathcal{B} \sim \mathcal{A}$, and

1). If $\mathcal{B}$ is uniform then so is $\mathcal{A}$.

2). If $\mathcal{B}$ is regular then so is $\mathcal{A}$.

3). $h(\mathcal{B}) = h(\mathcal{A})$.

4). If $\mathcal{B}$ is reaching height then so is $\mathcal{A}$.

Definition 4.1. If a fuzzy mapping $\tilde{f}: X \to \mathcal{F}(Y)$, $x \mapsto \tilde{f}(x)$ satisfies

$$\tilde{f}(xy) = \tilde{f}(x)\tilde{f}(y), \text{ for } \forall x, y \in X,$$

then $\tilde{f}$ is called a fuzzy homomorphism mapping from X to another group Y.

Theorem 4.2. Let $\tilde{f}: X \to \mathcal{F}(Y)$ be a fuzzy homomorphism mapping. If G is a subgroup of X, then

$$\mathcal{B} \triangleq \tilde{f}(G) \triangleq \{\tilde{f}(x) \mid x \in G\}$$

is a fuzzy HX group on Y and $G \sim \mathcal{B}$.

The proof is straight.

Theorem 4.3. Let $\tilde{f}: X \to \mathcal{F}(Y)$ be a fuzzy homomorphism mapping, f_λ be strong λ-cut mapping of $\tilde{f}$, where $\lambda \in [0,1]$. If G is a subgroup of X, then

$$f_\lambda(G) \triangleq \{f_\lambda(x) \mid x \in G\}$$

is a HX group on Y and $f_\lambda(G) \sim G$.

Theorem 4.4. Let $\tilde{f}: X \to \mathcal{F}(Y)$ be a fuzzy homomorphism mapping and $T_{\tilde{f}}$ be a fuzzy transformation guided by $\tilde{f}$ [2]. If $\mathcal{A}$ is a fuzzy HX group on X, then

$$\mathcal{B} \triangleq T_{\tilde{f}}(\mathcal{A}) \triangleq \{T_{\tilde{f}}(A) \mid A \in \mathcal{A}\}$$

is a fuzzy HX group on Y and $\mathcal{A} \sim \mathcal{B}$.

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