GREY FUZZY SETS

Chen Dawei and Mu Yizhong
Heilongjiang Water Conservancy Institute,
Harbin, China

Abstract: In this paper, the concept of grey fuzzy set(for short GF set) is practically given, thereby extends the fuzzy set. It is clarified that the grey number set is a typical instance of GF set.

Keywords: Direct product space, Grey fuzzy sets, Grey number, Membership.

The definition, operations and properties of GF set

We call unknown degree of not entirely known elements in some universe of discourse in grey system as the degree of point grey, and call the layer(or set) containing not entirely known elements as the grey layer(or grey set), otherwise, they are called 'no grey'. The following discussion is being had in the universe of double-layers(i.e. direct product space). And suppose the first layer is not grey.

Definition 1. Let the univeres $\Omega = X \times Y$ be direct product space, where element $\omega = (x, y)$ is the abstract point. The GF subset(for short GF set) in Ω is a class of the object which has both the degree of membership and the point grey degree, and its expression in mathematics is

$$\underset{\otimes}{\tilde{A}} = \tilde{A}_1 \times \underset{\otimes}{A}_2 = \{(y, \nu_{\underset{\otimes}{A}_2}(y)), p(y) \mid (x, \mu_{\tilde{A}_1}(x)), x \in X\}$$

When $\underset{\otimes}{\tilde{A}}$ is a finite discrete set (i.e. only containing finite elements), it can be written in enumeration method

$$\underset{\otimes}{\tilde{A}} = \{(y_1, \nu_{\underset{\otimes}{A}_2}(y_1)), (y_2, \nu_{\underset{\otimes}{A}_2}(y_2)) \ldots (y_n, \nu_{\underset{\otimes}{A}_2}(y_n)) \mid (x_1, \mu_{\tilde{A}_1}(x_1)), (x_2, \mu_{\tilde{A}_1}(x_2)), \ldots, (x_m, \mu_{\tilde{A}_1}(x_m))\}$$

where $\tilde{A}_1$ is the not grey Fuzzy set(for short F set) in space X, and $\underset{\otimes}{A}_2$ is the distinct grey set(for short G set) in space Y; $\mu_{\tilde{A}_1}(x)$ is the degree of membership of point x to $\tilde{A}_1$, $\nu_{\underset{\otimes}{A}_2}(y)$ is the point grey degree of point y to $\underset{\otimes}{A}_2$; $\mu_{\tilde{A}_1}$ and $\nu_{\underset{\otimes}{A}_2}$ are respectively mapping $X \to [0,1]$, $Y \to [0,1]$; $p(y)$ indicates the properties that point y posseses in $\underset{\otimes}{A}_2$, the greater is the point grey degree, the greater the unknown degree is. $\tilde{A}_1 \times \underset{\otimes}{A}_2$ are called $\underset{\otimes}{\tilde{A}}$ decomposition in X and Y. It is the direct product of both all points of making $\mu_{\tilde{A}_1}(x) \neq 0$ in X and all points of possibly possessed property $p(y)$ in Y.

According to definition 1, $\forall (x, y) \in \Omega$, when $\nu_{\underset{\otimes}{A}_2}(y) = 0$, $\underset{\otimes}{\tilde{A}} = \tilde{A}_1 \times A_2$ (i.e. $\underset{\otimes}{\tilde{A}}$ is white F set), when $\mu_{\tilde{A}_1}(x) \in \{0, 1\}$, $\underset{\otimes}{\tilde{A}} = A_1 \times \underset{\otimes}{A}_2$ is grey distinct set. If we take $\Omega = X$, then $\underset{\otimes}{\tilde{A}} = \tilde{A}_1 = \{(x, \mu_{\tilde{A}_1}(x)), x \in X\}$ is a general F set. This shows that GF set $\underset{\otimes}{\tilde{A}} = \tilde{A}_1 \times \underset{\otimes}{A}_2$ is the extension of F set indeed.

<u>Definition 2.</u> In $\Omega = X \times Y$, the entirety of GF set is called GF power set of Ω, which is written as $\mathcal{F}(\Omega)$.

<u>Definition 3.</u> If $\forall \underset{\otimes}{\tilde{A}} \in \mathcal{F}(\Omega)$, $\tilde{A}_1 = \phi$ or $\underset{\otimes}{A}_2 = \phi$, then $\underset{\otimes}{\tilde{A}}$ is called as empty GF set, written as ϕ.

<u>Definition 4.</u> If $\forall \underset{\otimes}{\tilde{A}}, \underset{\otimes}{\tilde{B}} \in \mathcal{F}(\Omega)$, we have

$$\mu_{\tilde{A}_1}(x) = \mu_{\tilde{B}_1}(x), \quad \nu_{\underset{\otimes}{A}_2}(y) = \nu_{\underset{\otimes}{B}_2}(y)$$

then $\underset{\otimes}{\tilde{A}}$ and $\underset{\otimes}{\tilde{B}}$ are called to be equal. We write as $\underset{\otimes}{\tilde{A}} = \underset{\otimes}{\tilde{B}}$.

In the cognition from people, any set is the unity of the connotation and extension for conception. Now we define the operations of the union, intersection and complement of GF sets respectively in the two following cases, the first layer $\tilde{A}_1$ is fuzzy (including distinct) while the second layer $\underset{\otimes}{A}_2$'s extension is grey (for short "EG") and $\underset{\otimes}{A}_2$'s connotation is grey (for short "CG").

I. <u>The union, intersection and complement of "EG" GF sets</u>

Because connotation decides extension, and whether extension is grey or not has nothing to do with the connotation being large or small, therefore, GF set's union and intersection could be defined as follows:

<u>Definition 5.</u> Let $\mathcal{F}(\Omega)$ be the power set of "EG" GF sets and $\forall \underset{\otimes}{\tilde{A}}, \underset{\otimes}{\tilde{B}} \in \mathcal{F}(\Omega)$,

$$\underset{\otimes}{\tilde{A}} = \{(y, \nu_{\underset{\otimes}{A}_2}(y)) : p_1(y) \mid (x, \mu_{\tilde{A}_1}(x)), x \in X\},$$

$$\underset{\otimes}{\tilde{B}} = \{(y, \nu_{\underset{\otimes}{B}_2}(y)) : p_2(y) \mid (x, \mu_{\tilde{B}_1}(x)), x \in X\},$$

then the union, intersection of $\underset{\otimes}{\tilde{A}}$ and $\underset{\otimes}{\tilde{B}}$ are defined as union $\underset{\otimes}{\tilde{A}} \cup \underset{\otimes}{\tilde{B}} \triangleq \{(y, \nu_{\underset{\otimes}{A_2}}(y) \vee \nu_{\underset{\otimes}{B_2}}(y)), p(y) \text{ or } p(y) \mid (x, \mu_{\tilde{A}_1}(x) \vee \mu_{\tilde{B}_1}(x)), x \in X\}$,

intersection $\underset{\otimes}{\tilde{A}} \cap \underset{\otimes}{\tilde{B}} \triangleq \{(y, \nu_{\underset{\otimes}{A_2}}(y) \wedge \nu_{\underset{\otimes}{B_2}}(y)), p_1(y) \text{ and } p_2(y) \mid (x, \mu_{\tilde{A}_1}(x) \wedge \mu_{\tilde{B}_1}(x)), x \in X\}$.

Because point grey degree of any GF set has nothing to do with the point grey degree of "GF set's complementary set", we may give the following definition:

Definition 6. Let $\mathcal{F}(\Omega)$ be the power set of "EG" GF set, and $\forall \underset{\otimes}{\tilde{A}} \in \mathcal{F}(\Omega)$,

$\underset{\otimes}{\tilde{A}} = \{(y, \nu_{\underset{\otimes}{A_2}}(y)), p(y) \mid (x, \mu_{\tilde{A}_1}(x)), x \in X\}$,

then the complement of $\underset{\otimes}{\tilde{A}}$ is defined as

$\underset{\otimes}{\tilde{A}}^{c} \triangleq \{(y, \nu_{Y-\underset{\otimes}{A_2}}(y)), \text{not } p(y) \mid (x, 1-\mu_{\tilde{A}_1}(x)), x \in X\}$,

where $Y-\underset{\otimes}{A_2}$ denotes the complementary set of $\underset{\otimes}{A_2}$ (comprehending it according to the complementary set of Cantor set).

Theorem 1. If $\mathcal{F}(\Omega)$ is the power set of "EG" GF sets, then the algebraic system $(\mathcal{F}(\Omega), \cup, \cap)$ is a distributive lattice containing 0 and 1, and $(\mathcal{F}(\Omega), \cup, \cap, {}^{c})$ is De Morgen algebra.

II. The union, intersection and complement of "CG" GF sets

Because of connotation being grey and not equal to connotation smill and large respectively, it defines the large and small of extension. Thus union and intersection can be defined as follows:

Definition 7. Let $\mathcal{F}(\Omega)$ be the power set of "CG" GF sets and $\forall \underset{\otimes}{\tilde{A}}, \underset{\otimes}{\tilde{B}} \in \mathcal{F}(\Omega)$,

$\underset{\otimes}{\tilde{A}} = \{(y, \nu_{\underset{\otimes}{A_2}}(y)), p_1(y) \mid (x, \mu_{\tilde{A}_1}(x), x \in X\}$,

$\underset{\otimes}{\tilde{B}} = \{(y, \nu_{\underset{\otimes}{B_2}}(y)), p_2(y) \mid (x, \mu_{\tilde{B}_1}(x), x \in X\}$,

then the two operations of "extension intersection and connotation union" (for short "EICU") and "extension union and connotation intersection" (for short "EUCI") for $\underset{\otimes}{\tilde{A}}$ and $\underset{\otimes}{\tilde{B}}$ are defined respectively as

EICU $\underset{\otimes}{\tilde{A}} \Cap \underset{\otimes}{\tilde{B}} \triangleq \{(y, \nu_{\underset{\otimes}{A_2}}(y) \vee \nu_{\underset{\otimes}{B_2}}(y)), p_1(y) \text{ or } p_2(y) \mid (x, \mu_{\tilde{A}_1}(x) \wedge \mu_{\tilde{B}_1}(x)), x \in X\}$,

EUCI $\underset{\otimes}{\tilde{A}} \Cup \underset{\otimes}{\tilde{B}} \triangleq \{(y, \nu_{\underset{\otimes}{A_2}}(y) \wedge \nu_{\underset{\otimes}{B_2}}(y)), p_1(y) \text{ and } p_2(y) \mid (x, \mu_{\tilde{A}_1}(x) \vee \mu_{\tilde{B}_1}(x)), x \in X\}$.

Definition 8. Let $\mathcal{F}(\Omega)$ be the power set of "CG" GF sets and $\forall \underset{\otimes}{\tilde{A}} \in \mathcal{F}(\Omega)$,

$\tilde{\underset{\otimes}{A}} = \{(y, \nu_{\underset{\otimes}{A}_2}(y)), p(y) \mid (x, \mu_{\tilde{A}_1}(x)), x \in X\}$,

then the complement of $\tilde{\underset{\otimes}{A}}$ is defined as

$\tilde{\underset{\otimes}{A}}^c \triangleq \{(y, \nu_{Y-\underset{\otimes}{A}_2}(y)), \text{not } p(y) \mid (x, 1-\mu_{\tilde{A}_1}(x)), x \in X\}$,

where $Y-\underset{\otimes}{A}_2$ denotes the complement set of $\underset{\otimes}{A}_2$ (It may be comprehended according to the complement set of Cantor set) .

Theorem 2. If $\mathcal{F}(\Omega)$ is the power set of "CG" GF sets, the algebraic system($\mathcal{F}(\Omega)$, ⋒, ⋓) will be a distributive lattice.

Grey number sets

Now concept of grey number set is introduced as an important special case of GF set.

Definition 9. Let both X, Y in the universe of discourse $\Omega = X \times Y$ be real number set R, and $\forall x \in R, \mu_{\tilde{A}_1}(x)$ equals 1 or 0, then

$\tilde{\underset{\otimes}{A}} = \underset{\otimes}{A} = A_1 \times \underset{\otimes}{A}_2 = \{(y, \nu_{\underset{\otimes}{A}_2}(y)), p(y) \mid (x, \mu_{A_1}(x)), x \in R\}$

is called grey number set, and its elements(x, y) are called Grey Number.

Example. Let $A_1 = \{\text{Odd Number}\}$, $\underset{\otimes}{A}_2 = \{\text{Prime Number}\}$, then

$\underset{\otimes}{A} = \{(y, \nu_{\underset{\otimes}{A}_2}(y)), y \text{ is a prime number} \mid (x, \mu_{A_1}(x)), x \in R\}$
$= \{\text{Odd Prime Number}\}$.

If $\forall x \in R$, $\mu_{A_1}(x) = 0$, then $\underset{\otimes}{A} = \phi$. Hence, we may only discuss the case of $\forall x \in R$, $\mu_{A_1}(x) = 1$, in this case, $A_1 = \{(x, 1), x \in R\}$ is a certain real number set, and it can be brought into the universe of discourse omitted without being written out, and $\underset{\otimes}{A}$ can now be written as $\{(y, \nu(y)), p(y)\}$, And according to the habit, we write x instead of y, thus we get

$\underset{\otimes}{A} = \{(x, \nu(x)), p(x)\}$

and can briefly denote as

$\underset{\otimes}{A} = R\{(x, \nu(x))\}$

or

$\underset{\otimes}{A} = R\{\otimes\}$

$\otimes$ on the right side of the equality is the general grey number [2] said by Deng Julong.

If the general grey number is denoted as

$\otimes = \otimes(x) = \underset{\otimes}{R}(x, \nu(x))$

then various grey numbers can be more concretely denoted by $\underset{\otimes}{R}(x_0, \nu(x_0))$, $\underset{\otimes}{R}(x_1, \nu(x_1))$, ..., $\underset{\otimes}{R}(x_i, \nu(x_i))$, ..., $\underset{\otimes}{R}(x_j, (x_j))$, ... Therefore, 'taking numbers consistent grey numbers 'said by Deng Julong may be written as

$$\underset{\otimes}{R}(x_i, (x_i)) = \underset{\otimes}{R}(x_j, (x_j)), \quad i \neq j .$$

If a certain grey number $\underset{\otimes}{R}(x_0, \nu(x_0))$ has whitened value[2] a, then it can be written

$$\underset{\otimes}{R}(x_0, 0) = a$$

and briefly denoted as

$$\otimes(x_0) = a \quad \text{or} \quad \otimes(a) = a .$$

<u>Definition 10</u>. GF set $\underset{\otimes}{A} = \{(x, \nu(x)), x \in [\underline{a}, \bar{a}] \text{ and } \underline{a}, \bar{a} \in R\}$ is called interval grey number set. Its elements are called interval grey numbers.

According to Deng Julong s representation, interval grey numbers can be written

$$\otimes \in [\underline{a}, \bar{a}]$$

where $\underline{a}$ and $\bar{a}$ are respectively called lower boundand upper bound of the interval grey numbers.

Similarly, {grey numbers having lower bound} and {grey numbers having upper bound} can also be defined. There is no need here to describe in detail.

People's cognition is infinite and is getting deeper and deeper along with layers. As we carry on investigations in objective things still further or more wide in order to make the "grey" of some characteristics of its connotation whitened, we may change its former membership (of course, we could also determine the former membership with even more assurance). Thereby some problems of "apparently right but actually wrong" can be finally solved. Just like fuzzy set theory could be applied to work out problems of "either this or that", generally speaking, for any thing whose connotation is grey, we may solve problems of "apparently right but actually wrong" through one of the operations of GF set. Undoubtedly, this is one type of an important application of GF set theory.

<u>References</u>

[1] L.A.Zadeh, Fuzzy Sets, Information and Control, 8(1965).
[2] Deng Julong, The Grey Controlling Systems, Central China Technology Institute, 1985.