ON THE EXISTENCE OF ONE-POINT ULTRA-FUZZY COMPACTIFICATIONS OF FUZZY NEIGHBORHOOD SPACES

Wolfgang Flüshöh
Fachbereich Mathematik
Bergische Universität-Gesamthochschule Wuppertal
Gaußstraße 20,5600 Wuppertal 1,Federal Republic of Germany

In terms of the theory of ordinary compact topological spaces we give a necessary and sufficient condition for a fuzzy neighborhood space to possess a one-point ultra-fuzzy compactification.

First of all, in section 0 we collect some definitions and facts which will be needed in the sequel.
In section 1 we describe a method of constructing a one-point ultra-fuzzy quasi-compactification for an arbitrary fuzzy neighborhood space.
By the aid of this one-point ultra-fuzzy quasi-compactification in section 2 we give a necessary and sufficient condition for a fuzzy neighborhood space to possess a one-point ultra-fuzzy compactification.
Finally we present an example of a fuzzy neighborhood space not possessing a one-point ultra-fuzzy compactification.

0. Preliminaries

S denotes a nonvoid set and **P**(S) the power set of S.

For the definition and the fundamental properties of fuzzy neighborhood spaces the reader is referred to [5].

If (S,Δ) is a fuzzy neighborhood space with associated fuzzy neighborhood system $(U_p)_{p\in S}$ in S and α is an element of $[0,1[$ then the family $\iota_\alpha((U_p)_{p\in S}) := (U(p,\alpha))_{p\in S}$ given by

$$p\in S:\ U(p,\alpha) := \{U\in \mathbf{P}(S) \mid \exists f\in U_p,\ \exists \beta\in]0,1-\alpha[:\ \{f\geq\beta\} \subseteq U\}$$

is the ordinary neighborhood system in S defining the α-level-topology of (S,Δ). The associated topological space will be denoted (S,Δ_α).
Moreover, $\iota_0((U_p)_{p\in S})$ is the ordinary neighborhood system in S belonging to the 1-topology of (S,Δ) (cf. [3]).

In this connection we need the following result:

0.1 Proposition: (cf. [1]) The relation

$$(U_p)_{p\in S} \mapsto (\iota_\alpha((U_p)_{p\in S}))_{\alpha\in]0,1[}$$

defines a bijective mapping from the set of all fuzzy neighborhood systems $(U_p)_{p\in S}$ in S onto the set of all]0,1[-indexed families $((U(p,\alpha))_{p\in S})_{\alpha\in]0,1[}$ of ordinary neighborhood systems $(U(p,\alpha))_{p\in S}$ in S provided with the property

(NC) $\quad U(p,\alpha) = \bigcup_{\beta\in]\alpha,1[} U(p,\beta)$ for every $\alpha\in]0,1[$; $p\in S$.

In particular, for every fuzzy neighborhood space (S,Δ) the subsequent conditions are equivalent:

(i) (S,Δ) is topologically generated (cf. [3])

(ii) For some ordinary topology $\mathfrak{T}$ on S we have:

$$\Delta_\alpha = \mathfrak{T} \quad \text{for every } \alpha\in]0,1[\; .$$

In this case, Δ is said to be topologically generated by $\mathfrak{T}$ (cf. [3]).

0.2 Proposition: Let (S,Δ) and (T,Θ) be fuzzy neighborhood spaces. Then for every mapping $a:S \to T$ the following conditions are equivalent:

(i) a is Δ-Θ-continuous (cf. [5],[4])

(ii) a is Δ_α-Θ_α-continuous for every $\alpha\in]0,1[$.

In particular, if $S \subseteq T$ then the subsequent conditions are equivalent:

(i) (S,Δ) is a subspace of (T,Θ) (cf. [4])

(ii) (S,Δ_α) is a subspace of (T,Θ_α) for every $\alpha\in]0,1[$.

In the sequel (S,Δ) denotes a fuzzy neighborhood space with related fuzzy neighborhood system $(U_p)_{p\in S}$ in S. Further, for every $\alpha\in]0,1[$, $H_\alpha:\mathbf{P}(S) \to \mathbf{P}(S)$ denotes the ordinary closure operator belonging to the α-level-topology of (S,Δ).
Then (NC) is equivalent to the condition

(CC) $\quad H_\alpha(A) = \bigcap_{\beta\in]\alpha,1[} H_\beta(A)$ for every $\alpha\in]0,1[$; $A\in\mathbf{P}(S)$.

Moreover, the closure operator H_0 related to the 1-topology of (S,Δ) is given by:

$$H_0(A) = \bigcap_{\alpha\in]0,1[} H_\alpha(A) \quad \text{for every } A\in\mathbf{P}(S) \; .$$

0.3 Definition: a) (cf. [2],[6]) (S,Δ) is said to be ultra-fuzzy quasi-compact (ultra-fuzzy compact) iff the 1-topology of (S,Δ) is quasi-compact (compact; i.e. quasi-compact and Hausdorff-separated).

b) A fuzzy neighborhood space $(\hat{S},\hat{\Delta})$ is called a one-point ultra-fuzzy quasi-compactification (ultra-fuzzy compactification) of (S,Δ) iff the following conditions are satisfied:

(i) $(\hat{S},\hat{\Delta})$ is ultra-fuzzy quasi-compact (ultra-fuzzy compact)

(ii) (S,Δ) is homeomorphic with a subspace (T,Θ) of $(\hat{S},\hat{\Delta})$, where the complement $\complement_{\hat{S}}T$ of T in $\hat{S}$ contains exactly one point .

1. A one-point ultra-fuzzy quasi-compactification

Let Γ denote the set of all Δ_0-relatively-quasi-compact subsets of S and for every $\alpha\in]0,1[$ let the subset $Q(\alpha)$ of $\mathbf{P}(S)$ be given by:

$$Q(\alpha) := \{A\in\mathbf{P}(S) \mid \exists\, \beta\in]\alpha,1[: H_\beta(A)\in\Gamma\} .$$

1.1 Lemma: $((Q(\alpha))_{\alpha\in]0,1[}$ possesses the properties:

(QS1) $\alpha\in]0,1[: \emptyset\in Q(\alpha)$

(QS2) $\alpha\in]0,1[: A_1,A_2\in Q(\alpha) \Rightarrow A_1\cup A_2\in Q(\alpha)$

(QS3) $\alpha\in]0,1[: A\in Q(\alpha) \Rightarrow H_\alpha(A)\in Q(\alpha)$

(QS4) $\alpha\in]0,1[: Q(\alpha) = \bigcup_{\beta\in]\alpha,1[} Q(\beta)$.

Proof: In view of (CC) the assertion follows immediately from the definitions.

Let ω be any object not belonging to S and define $\hat{S} := S\cup\{\omega\}$. Further, for every $\hat{p}\in\hat{S}$ and every $\alpha\in]0,1[$ let the subset $\hat{U}(\hat{p},\alpha)$ of $\mathbf{P}(\hat{S})$ be defined according to:

$$\hat{U}(\hat{p},\alpha) := \begin{cases} \{\hat{U}\in\mathbf{P}(\hat{S}) \mid \exists\, U\in U(p,\alpha):\ U\subseteq\hat{U}\}, & \text{if } \hat{p}=p\in S \\ \{\hat{U}\in\mathbf{P}(\hat{S}) \mid \exists\, A\in Q(\alpha):\ \{\omega\}\cup\complement_S A\subseteq\hat{U}\}, & \text{if } \hat{p}=\omega . \end{cases}$$

1.2 Lemma: For every $\alpha\in]0,1[$, $(\hat{U}(\hat{p},\alpha))_{\hat{p}\in\hat{S}}$ is the neighborhood-system belonging to the α-level-topology of a fuzzy neighborhood space $(\hat{S},\hat{\Delta})$.

(S,Δ) is a subspace of $(\hat{S},\hat{\Delta})$.

Proof: From (QS1), (QS2) and (QS3) we infer that $(\hat{U}(\hat{p},\alpha))_{\hat{p}\in\hat{S}}$ is an ordinary neighborhood system in $\hat{S}$. (QS4) implies (NC) and therewith the first assertion follows from Proposition 0.1 .
In view of Proposition 0.2 the second assertion is obvious.

1.3 Proposition: $(\hat{S},\hat{\Delta})$ is a one-point ultra-fuzzy quasi-compactification of (S,Δ).

Proof: To prove that $(\hat{S},\hat{\Delta}_0)$ is quasi-compact let $\mathbf{U}$ be an ultrafilter on $\hat{S}$. Assume that $\mathbf{U}$ is not $\hat{\Delta}_0$-convergent to some $p\in S$, let α be an element of $]0,1[$ and let $\hat{U}$ be a $\hat{\Delta}_0$-neighborhood of ω. Then for some $A\in\mathbf{P}(S)$, $\beta\in]\alpha,1[$ we have $\complement_{\hat{S}}A\le\hat{U}$ and $H_\beta(A)\in\Gamma$. Since $H_0(A)\le H_\beta(A)$ this implies $\complement_{\hat{S}}H_0(A)\in\mathbf{U}$. Now we infer that $\mathbf{U}$ is convergent to ω.

1.4 Remark: If Δ is topologically generated by $\mathcal{T}$ then $\hat{\Delta}$ is topologically generated by the Alexandroff-quasi-compactification of $\mathcal{T}$ (cf. [7]).

2. One-point ultra-fuzzy compactifications

2.1 Proposition: The following conditions are equivalent:

(i) (S,Δ) possesses a one-point ultra-fuzzy compactification

(ii) (S,Δ_0) is Hausdorff-separated and
$\forall\ A\in\Gamma\ \exists\ \alpha\in]0,1[$ so that $H_\alpha(A)\in\Gamma$.

Proof: (i)=>(ii): Let $(\tilde{S},\tilde{\Delta})$ be a one-point ultra-fuzzy compactification of (S,Δ), where $\tilde{S}=S\cup\{\tilde{\omega}\}$.
In particular, we may assume that $(\tilde{S},\tilde{\Delta}_0)$ is the Alexandroff-compactification of (S,Δ_0).
Choose $A\in\Gamma$. Then $U:=\{\tilde{\omega}\}\cup\complement_S A$ is a $\tilde{\Delta}_0$-neighborhood of $\tilde{\omega}$ and we can find $\alpha\in]0,1[$ so that U is a $\tilde{\Delta}_\alpha$-neighborhood of $\tilde{\omega}$.
Let V be a $\tilde{\Delta}_\alpha$-neighborhood of $\tilde{\omega}$ so that U is a $\tilde{\Delta}_\alpha$-neighborhood of q

for every q in V. Since V is a $\tilde{\Delta}_0$-neighborhood of $\tilde{\omega}$ we can find B in Γ so that $\{\tilde{\omega}\}\cup\complement_S B \subseteq V$. Now we obtain $H_\alpha(A) \subseteq B$, i.e. $H_\alpha(A)\in\Gamma$.
(ii)⇒(i): For the one-point ultra-fuzzy quasi-compactification $(\hat{S},\hat{\Delta})$ of (S,Δ) we have: $\hat{U}(\omega,0) = \{\hat{U}\in P(\hat{S}) \mid \exists A\in\Gamma: \{\omega\}\cup\complement_S A \subseteq \hat{U}\}$;
i.e. $(\hat{S},\hat{\Delta}_0)$ is the Alexandroff-compactification of (S,Δ_0).

An example of a fuzzy neighborhood space not possessing a one-point ultra-fuzzy compactification:

2.2 Example: Let **R** denote the set of real numbers and for every $\alpha\in]0,1[$ define $H_\alpha: P(\mathbf{R}) \to P(\mathbf{R})$ according to:

$$A\in P(\mathbf{R}): \quad H_\alpha(A) := \begin{cases} \emptyset, & \text{if } A=\emptyset \\ A\cup[\frac{1}{\alpha},\infty[, & \text{otherwise} \end{cases} .$$

Then $(H_\alpha)_{\alpha\in]0,1[}$ fulfills the condition (CC) and thus defines in view of Proposition 0.1 a fuzzy neighborhood space (S,Δ).
(S,Δ) possesses a locally compact 1-topology but no one-point ultra-fuzzy compactification.

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