FUZZY MAPPINGS AND FUZZY PRODUCTION MODEL

Part 2: Fuzzy production model

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This part is devoted to a fuzzy production model and to the optimal trajectories.

3. Description of the model.

In this section we will deal with the description of a fuzzy model of economic dynamics. Let us recall that in a classical model the technology of an economy in a time interval (e.g. from a moment t to moment τ) is described by a multifunction. In this way, a certain set of commodity bundles is assigned to a certain commodity bundle.

A fuzzy model of economic dynamics is an object

$$M = \{ E, (R^{n_t})_{t \in E}, (K_t)_{t \in E}, (f_{t\tau})_{(t,\tau) \in \widetilde{E}} \} ,$$

where

- $E = \{ t \in R : t \geq 0 \}$,
- $\widetilde{E} = \{ (t, \tau) \in E \times E : t < \tau \}$,
- R^{n_t} - n_t - dimensional Euclidean space,
- $R^{n_t} \supset K_t$ - a convex cone,
- $f_{t\tau}$ - superadditive, positive homogeneous and closed fuzzy mapping, $f_{t\tau} : K_t \to L(K_\tau)$.

It is assumed that the class $(f_{t\tau})_{(t,\tau) \in \widetilde{E}}$ has the following property: if $t, \tau, \theta \in E$ and $t < \tau < \theta$ then $f_{t\theta} = f_{\tau\theta} \circ f_{t\tau}$. Elements of E will be called time moments, and the element $0 \in E$ initial time moment.

A technological trajectory of M is a family $Tr = (x_t)_{t \in E}$ such that:

- $x_t \in K_t$, $(t \in E)$,
- $f_{t\tau}^{x_t}(x_\tau) > 0$, $((t, \tau) \in \widetilde{E})$.

In this case x_t is called the state of the trajectory Tr at the time t, x_0 is the initial state of Tr. It can be seen that in the case of a technological trajectory it is possible to determine at each stage the producer's valuation of the authenticity of the obtained commodity bundle. Without impairing the general character of our considerations it can be assumed that we are not interested here in any commodity bundles, but only those whose producers valuation of the authenticity is not lower than the beforehand fixed level.

Now we will formulate a theorem of the existence of a technological trajectory in our model.

Theorem (Existence of technological trajectory). Let $0, t' \in E$ such that $0 < t'$, $y_0 \in K_0$ and $f_{0t'}^{y_0}(y_{t'}) > 0$. Then there exists a technological trajectory of M going out y_0 and passing through $y_{t'}$ at the moment of time t'.

Proof. For any $(t, \tau) \in \tilde{E}$ and any $x_t \in K_t$ let us define a set

$$a_{t\tau}(x_t) = \{ x_\tau : f_{t\tau}^{x_t}(x_\tau) > 0 \}.$$

In this way a fuzzy mapping $f_{t\tau}$ generate a multifunction $a_{t\tau}$. From the above assumptions of the class $(f_{t\tau})_{(t,\tau)\in\tilde{E}}$ it follows that the class $(a_{t\tau})_{(t,\tau)\in\tilde{E}}$ has the following properties:

- for any $(t, \tau) \in \tilde{E}$

  1° $a_{t\tau}$ is positive homogeneous, superadditive and closed multifunction.

  2° if $t < \tau < \theta$ then $a_{t\theta} = a_{\tau\theta} \circ a_{t\tau}$.

Now, using the methods of classical analysis we may prove the theorem For the construction of this proof in classical case see [4].

4. Optimal technological trajectories.

Now, let us additionally assume that there exists $T \in E$ such that $t \leq T$ for all $t \in E$.

Definition 4.1. A technological trajectory with initial state x_0 and terminal state x_T is called optimal if

$$f_{OT}^{x_0}(x_T) = \sup_{y \in R^{n_T}} f_{OT}^{x_0}(y) .$$

We see that the sence of optimality means the choice of such an end state which has the highest producer's valuation of the authenticity of all the states that can be derived from a given initial state.

Now, let us define the fuzzy subset $G^{x_T} \subset R^{n_T}$ such that

$$G^{x_T}(y) = \begin{cases} f_{OT}^{x_0}(y) & \text{if } y = x_T , \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.1. A technological trajectory Tr with initial state x_0 and terminal state x_T is optimal iff the separation degree of the fuzzy subsets $f_{OT}^{x_0}$ and G^{x_T} is equal

$$1 - \sup_{x \in R^{n_T}} f_{OT}^{x_0}(x) .$$

Proof. Let Tr denote an optimal trajectory with initial and terminal states x_0 and x_T respectively. From the definition 4.1 and theorem 2.4 it follows that the fuzzy subsets $f_{OT}^{x_0}$ and G^{x_T} are bounded and convex. Therefore their separation degree is equal

$$1 - \sup_{x \in R^{n_T}} (f_{OT}^{x_0}(x) \wedge G^{x_T}(x)) = 1 - f_{OT}^{x_0}(x_T) =$$

$$= 1 - \sup_{x \in R^{n_T}} f_{OT}^{x_0}(x).$$

Let now the separation degree of fuzzy subsets $f_{0T}^{x_o}$ and G^{x_T} be equal to $1 - \sup_{x \in R^{n_T}} f_{0T}^{x_o}(x)$. So, in accordance with the Separation Theorem for fuzzy subsets we have

$$\sup_{x \in R^{n_T}} f_{0T}^{x_o}(x) = \sup_{x \in R^{n_T}} (f_{0T}^{x_o}(x) \wedge G^{x_T}(x)) = f_{0T}^{x_o}(x_T).$$

Therefore the trajectory Tr is optimal.

Definition 4.2. A technological trajectory Tr with initial and terminal states x_o and x_T respectively is called (r,p)-optimal if there exists a non-zero functional $p \in K_T^{*}$ such that

$$p(x_T) = \max_{x \in (f_{0T}^{x_o})^r} p(x) > 0, \qquad (*)$$

where

$$(f_{0T}^{x_o})^r = \{ x : f_{0T}^{x_o}(x) \geqslant r \}.$$

In this instance, functional p can be regarded as commodity prices. However, the commodity bundle value at given prices is not complete information, but it is supplemented by the producer's valuation of authenticity of this commodity bundle. This valuation fulfills an important function when we deal with commodity bundles of similar value but various of producer's valuations of authenticity. In the above definition we consider such the terminal states for which the producer's valuations of authenticity is not less than those fixed beforhand. For instance, if r denote the lowest producer's valuation of authenticity of considered commodity bundles which we can obtaine from the state x_o then $(f_{0T}^{x_o})^r$ is a set of such commodity bundles.

Let A denote a subset of R^{n_T} ($A \neq \emptyset, \{0\}$). An element x of A is called the limiting point from above of A if $a \cdot x \notin A$ for $a > 1$.

For a normal covering of A we use the symbol nA (compare [4]).

Theorem 4.2. A technological trajectory Tr with initial state x_0 and terminal state x_T is (r,p)-optimal iff the element x_T is a limiting point from above of the set $n(f_{0T}^{x_0})^r$.

Proof. Let Tr denote a (r,p)-optimal trajectory with initial and terminal states x_0 and x_T respectively. We will prove that x_T (with $F = f_{0T}^{x_0}$ for convenience) is a limiting point from above of F^r. In contrary, suppose there exists an $a > 1$ such that $a \cdot x_T \in nF^r$. Because $p(x_T) > 0$ we get

$$p(x_T) = \max_{x \in F^r} p(x) = \max_{x \in nF^r} p(x) \geqslant p(a\, x_T) = a \cdot p(x_T) > 0,$$

i.e. $1 \geqslant a$ in contradiction with $a > 1$.

Now, let us assume that x_T is a limiting point from above of the set nF^r. Let S denote the sphere $nF^r - nF^r$ and $\|\cdot\|_{nF^r}$ Minkowski's norm. It is known that this norm is monotonous and

$$nF^r = \{ z : \|z\|_{nF^r} \leqslant 1 \}.$$

Because x_T is a limiting point from above of nF^r, there holds

$$\| x_T \|_{nF^r} = 1.$$

Therefore, there exists a functional $p \in K_T^*$ such that

$$p(x_T) = \| x_T \|_{nF^r} = 1 , \quad \| p \| = 1 .$$

So, it is proved that for the trajectory Tr the condition (*) is fulfilled and thus the proof is finished.

References

[1] Butnariu, D., Fixed points for fuzzy mappings, Fuzzy Sets and Systems 7, 191-207, 1982.

[2] Debreu, G., Theory and value, John Wiley, New York, 1959.

[3] Lancaster, K., Mathematical Economics, New York, 1968.

[4] Makarov, V.L., and Rubinov A.M., Mathematical Theory of economic dynamic and equilibrium, Moskva, 1973 (in Russian).

[5] Morishima, M., Equilibrium stability and growth, Oxford University Press, 1964.

[6] Nikaido, H., Convex structures and economic theory, Academic Press, 1968.

[7] Ponsard, C., An application of fuzzy subsets theory to the analysis of the consumer's spatial preferences, Fuzzy Sets and Systems 5, 235-244, 1981.

[8] Ponsard, C., On the imprecision of consumer's spatial preferences, Papers Regional Science Association 42, 59-71, 1979.

[9] Ponsard, C., Fuzzy economic spaces, Working Paper of the I.M.E. presented at the First World Regional Science Congress, Cambridge , 1980.

[10] Ponsard, C., L'Equilibre Spatial du Consommateur dans un Contexte Imprecis, Sistemi Urbani, 1-2, 107-133, 1981.

[11] Ponsard, C., Producer's spatial equilibrium with fuzzy constaint, European Journal of Operational Research 10, 302-313, 1982.

[12] Ponsard, C., Partial spatial equilibria with fuzzy constraints, Journal of Regional Science 22, 1982.

[13] Zadeh, L.A., Fuzzy sets, Information and Control 8, 338-353, 1965.

[14] Zadeh, L.A., The concept of a linguistic variable and its applicationto approximate reasoning. Part I, Information Sciences 8, 199-249, 1975.