SOME REMARKS ON THE TOPOLOGICAL PROPERTIES OF FUZZY NUMBERS IN $\mathbb{R}^n$

L. GERGÓ

Computer Centre of Eötvös Loránd University,
1117 Budapest, Bogdánfy u. 10/b

Abstract: The space of normal, upper semicontinuous, fuzzy convex and compactly supported fuzzy numbers in $\mathbb{R}^n$ are considered endowed with different metrics generated by the Hausdorff metric. A subspace is introduced and properties of the metrics restricted to the subspace are investigated.

Keywords: fuzzy numbers, metric space, Hausdorff metric, equivalence of metrics, product of fuzzy numbers

1. Introduction

Many authors (see for example: Kaleva [1],[2],[3] Diamond, Kloeden [4] ; Puri, Ralescu [5],[6] ; Goetschel, Woxman [7]) deal with the convenient metric space of normal, upper semicontinuous, fuzzy convex and compactly supported fuzzy numbers in $\mathbb{R}^n$. So it is important to know more about this space endowed with different metrics. We will show that the metrics on the set of fuzzy numbers defined in $\mathbb{R}^n$ are equivalent to those which we get as a product of metrics on the set of fuzzy numbers in $\mathbb{R}$.

2. Notations

Denote E^n the set of all normal (i.e. there exists $t_o \in R^n$ such that $x(t_o)=1$) , fuzzy convex, upper semicontinuous and compactly

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supported fuzzy numbers in R^n , where fuzzy convex means that for the function $x:R^n \to I$

$$x(\alpha t+(1-\alpha)s) \geq \min\{x(t),x(s)\}$$

holds for each $t,s \in supp(x)$.

Define the metric D_n by the equation

$$D_n(x,y) = \sup_{\alpha \in I} d_n([x]^\alpha,[y]^\alpha)$$

where I denotes the closed interval $[0,1]$,

d_n is the Hausdorff metric in $P_K(R^n)$,

$d_n(K,L)=\max\{\sup_{x\in K} \rho(x,L),\sup_{y\in L} \rho(K,y)\}$, where $\rho(x,L)$ denotes the ρ-distance of the point x and the subset L in $\mathbb{R}^n$.

$[x]^\alpha = \{t\in R^n \mid x(t)\geq\alpha\}$ for $0<\alpha\leq 1$, the α-level set of x .

$[x]^0$ denotes the support of x .

It is known that the α-level sets of x are nonempty convex compact subsets of R^n and the space (E^n,D) is a complete metric space .

3. Equivalence of Hausdorff metrics

For a fixed metric ρ we can give an equivalent definition for the Hausdorff metric d_n in $P_K(\mathbb{R}^n)$ (see [7])

$$d_n(K,L) := \inf \varepsilon_\rho(K,L)$$

where $\varepsilon_\rho(K,L) = \{ \varepsilon>0 : K\subset(L)^\rho_\varepsilon$ and $L\subset(K)^\rho_\varepsilon \}$. Here $(L)^\rho_\varepsilon$ denotes the paralel domain of the set L with respect to the metric ρ that is

$$(L)^\rho_\varepsilon = \{ x\in\mathbb{R}^n : \rho(x,L)\le\varepsilon\}$$

Let us consider the metric spaces $(\mathbb{R}^n,\rho)$ and $(\mathbb{R}^n,\rho')$. Suppose that there exist positive constants c_1, c_2 such that

$$c_1 \rho' \le \rho \le c_2 \rho'$$

Question: what can we say about the two corresponding Hausdorff metrics d and d'.

Lemma 3.1 For the metrics ρ and ρ' given as above and for each $K\in P_K(\mathbb{R}^n)$ the following holds

$$\left(K\right)^{\rho'}_{\varepsilon/c_2} \subset \left(K\right)^{\rho}_{\varepsilon} \subset \left(K\right)^{\rho'}_{\varepsilon/c_1}$$

Proof It follows at once from the equvivalence of the metrics and the definition of $(K)^\rho_\varepsilon$.

Lemma 3.2 For given metrics ρ and ρ' in $P_K(\mathbb{R}^n)$ the Hausdorff metrics d and d' belonging to them , respectively , fulfil the inequalities

$$c_1 d' \le d \le c_2 d'$$

that is the Hausdorff metrics in $P_K(\mathbb{R}^n)$ are equivalent.

Proof

$$\varepsilon \geq d(K,L) \Rightarrow \varepsilon \in \varepsilon_\rho(K,L) \Rightarrow K \subset (L)^\rho_\varepsilon \text{ and } L \subset (K)^\rho_\varepsilon$$

by the Lemma 3.1 we obtain that

$$K \subset \left[L\right]^{\rho'}_{\varepsilon/c_1} \text{ and } L \subset \left[K\right]^{\rho'}_{\varepsilon/c_1} \Rightarrow \frac{\varepsilon}{c_1} \in \varepsilon_{\rho'}(K,L) \Rightarrow \frac{\varepsilon}{c_1} \geq d'(K,L)$$

$$\Rightarrow \varepsilon \geq c_1 \cdot d'(K,L) \quad \text{that is} \quad d(K,L) \geq c_1 \cdot d'(K,L) \; .$$

Conversely

$$\varepsilon \geq d'(K,L) \Rightarrow \varepsilon \in \varepsilon_{\rho'}(K,L) \Rightarrow K \subset (L)^{\rho'}_\varepsilon \text{ and } L \subset (K)^{\rho'}_\varepsilon$$

by the Lemma 3.1 we obtain that

$$K \subset \left[L\right]^{\rho}_{\varepsilon c_2} \text{ and } L \subset \left[K\right]^{\rho}_{\varepsilon c_2} \Rightarrow \varepsilon \cdot c_2 \in \varepsilon_\rho(K,L) \Rightarrow \varepsilon \cdot c_2 \geq d(K,L)$$

$$\Rightarrow \varepsilon \geq \frac{1}{c_2} d(K,L) \Rightarrow c_2 \cdot d'(K,L) \geq d(K,L)$$

■

Definition 3.1. The product $x = x_1 \times x_2 \times \ldots \times x_n$ of the fuzzy numbers $x_1, x_2, \ldots, x_n \in E^1$ is the following element of E^n .

$$x(t) = \min \{x_1(t_1), x_2(t_2), \ldots, x_n(t_n)\} \quad \text{for each} \quad t \in \mathbb{R}^n \; .$$

Denote $\mathcal{E}^n$ the subset of E^n with the definition

$$\mathcal{E}^n := \{x_1 \times x_2 \times \ldots \times x_n \in E^n : \; x_i \in E^1 \text{ for each } 1 \leq i \leq n\}$$

Lemma 3.3. For each $x \in \mathcal{E}^n$

$$[x]^\alpha = [x_1]^\alpha \times [x_2]^\alpha \times \ldots \times [x_n]^\alpha$$

Proof

$$t \in [x]^\alpha \Leftrightarrow x(t) \geq \alpha \Leftrightarrow \min\{x_1(t_1), x_2(t_2), \ldots, x_n(t_n)\} \geq \alpha$$

$$\Leftrightarrow x_i(t_i) \geq \alpha \text{ for each } 1 \leq i \leq n \Leftrightarrow t_i \in [x_i]^\alpha \text{ for each } 1 \leq i \leq n$$

$$\Leftrightarrow \quad t \in [x_1]^\alpha \times [x_2]^\alpha \times \ldots \times [x_n]^\alpha$$

■

4. Restriction of the metrics D_n and D_n^p to the subspace $\mathcal{E}^n$

In this paragraph we are interested in the question: what will happen if we restrict the metrics defined in the space E^n to the subspace $\mathcal{E}^n$ which consists of the product-form elements of E^n.

Now we fix the Hausdorff metric d_n which is defined by the maximum metric in $\mathbb{R}^n$. If we consider the subset of the n-dimensional compact cubes in $\mathbb{R}^n$ we have the following simple relation for the paralel domain of a cube.

$$(*) \qquad (I^n)_\rho = (I_1)_\rho \times (I_2)_\rho \times \ldots \times (I_n)_\rho$$

where $I^n = I_1 \times I_2 \times \ldots \times I_n$.

Of course, if we used another metric in $\mathbb{R}^n$, the formula above would not hold.

Theorem 4.1. For each n-dimensional cubes $I^n, J^n \in \mathcal{P}_K(\mathbb{R}^n)$ we have the following

(i) $$d_n(I^n,J^n) = \max_{1\le k\le n} d_1(I_k,J_k)$$

(ii) $$\frac{1}{n}\cdot\sum_{k=1}^{n} d_1(I_k,J_k) \le d_n(I^n,J^n) \le \sum_{k=1}^{n} d_1(I_k,J_k)$$

Proof It is enough to prove part (i) because part (ii) is a straightforward consequence of (i) .

For an arbitrary positive number ρ , $\rho\ge \max_{1\le k\le n} d_1(I_k,J_k)$ if and only if $\rho\ge d_1(I_k,J_k)$ for each $1\le k\le n$. By the definition of the Hausdorff metric it is equivalent to the fact that

$$(I_k)\subset(J_k)_\rho \quad\text{and}\quad (J_k)\subset(I_k)_\rho \quad\text{for}\quad k\in\{1,2,\dots,n\}$$

Using formula (*)

$$I^n\subset(J^n)_\rho \quad\text{and}\quad J^n\subset(I^n)_\rho$$

that is

$$\rho\ge d_n(I^n,J^n)$$

This means that

$$\rho\ge \max_{1\le k\le n} d_1(I_k,J_k) \quad\text{if and only if}\quad \rho\ge d_n(I^n,J^n)$$

this completes the proof.

■

Theorem 4.2. For each $x,y\in\mathcal{E}^n$,

(i) $$D_n(x,y) = \max_{1\le k\le n} D_1(x_k,y_k)$$

$$\text{(ii)} \qquad k_1 \cdot \left[\sum_{k=1}^{n} (D_1(x_k,y_k))^p\right]^{\frac{1}{p}} \le D_n^p(x,y) \le k_2 \cdot \left[\sum_{k=1}^{n} (D_1(x_k,y_k))^p\right]^{\frac{1}{p}}$$

for some positive constants k_1, k_2 .

Proof

$$\text{(i)} \qquad D_n(x,y) = \sup_{\alpha \in I} d_n([x]^\alpha,[y]^\alpha) = \sup_{\alpha \in I} \max_{1 \le k \le n} d_1([x_k]^\alpha,[y_k]^\alpha) =$$

$$= \max_{1 \le k \le n} \sup_{\alpha \in I} d_1([x_k]^\alpha,[y_k]^\alpha) = \max_{1 \le k \le n} D_1(x_k,y_k)$$

$$\text{(ii)} \qquad D_n^p(x,y) = \left[\int_I (d_n([x]^\alpha,[y]^\alpha))^p \, d\alpha\right]^{\frac{1}{p}} =$$

$$= \left[\int_I (\max_{1 \le k \le n} d_1([x_k]^\alpha,[y_k]^\alpha))^p \, d\alpha\right]^{\frac{1}{p}} \le \left[\int_I (c_2 \cdot \sum_{k=1}^{n} d_1([x_k]^\alpha,[y_k]^\alpha))^p \, d\alpha\right]^{\frac{1}{p}}$$

$$\le \left[\int_I (c_2^p \cdot n^{n-1} \cdot \sum_{k=1}^{n} d_1([x_k]^\alpha,[y_k]^\alpha))^p\right]^{\frac{1}{p}} =$$

$$= c_2 \cdot \sqrt[p]{n^{n-1}} \left[\int_I \sum_{k=1}^{n} (d_1([x_k]^\alpha,[y_k]^\alpha))^p \, d\alpha\right]^{\frac{1}{p}} =$$

$$= c_2 \cdot \sqrt[p]{n^{n-1}} \left[\sum_{k=1}^{n} \int_I (d_1([x_k]^\alpha, [y_k]^\alpha))^p \, d\alpha \right]^{\frac{1}{p}} =$$

$$= c_2 \cdot \sqrt[p]{n^{n-1}} \left[\sum_{k=1}^{n} (D_1(x_k, y_k))^p \right]^{\frac{1}{p}}$$

The other side of the inequalities can be obtained in the same way. This theorem shows the interesting fact that the restrictions of the metrics D_n and D_n^p result in these product metrics.

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