SOME EXTENSIONS OF NGUYEN'S THEOREM*

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The goal of this paper to generalize certain results of Nguyen [1] (concerning the α-cuts of two-place functions defined by the Zadeh's extension principle) to context of extended two-place functions defined via a sup- t-norm convolution.

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1. Introduction

In this paper, generalizing Nguyen's theorem, we give a necessary and sufficient condition for obtaining the equality

$$[f(A,B)]_\alpha = \bigcup_{T(\xi,\eta)\geq\alpha} f(A_\xi, B_\eta) \qquad \alpha\in(0,1] \qquad (*)$$

where $f:X\times Y\to Z$, T is a t-norm, A and B are fuzzy subsets of X and Y, respectively, $f(A,B)$ is defined via sup- T-norm convolution, A_α and B_α are the α-level sets of A and B, respectively, and $[f(A,B)]_\alpha$ is the α-level set of $f(A,B)$.

Furthermore, we shall define a class of fuzzy subsets in which this equality holds for all upper semicontinuous (u.s.c.) T and continuous f.

It should be noted that in the special case $T(x,y)=\min\{x,y\}$, the equation (*) yields

$$[f(A,B)]_\alpha = f(A_\alpha, B_\alpha) \qquad \alpha\in(0,1]$$

which coincides with Nguyen's result.

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2. Definitions

The symbol $\mathcal{F}(X)$ denotes the family of all fuzzy subsets of a set X. Let X be a topological space and denote by $\mathcal{F}(X,\mathcal{X})$ the set of all fuzzy subsets of X having u.s.c., compactly-supported membership function.

The support of a fuzzy set $A\in\mathcal{F}(X)$ is defined by

$$\mathrm{supp}A = \mathrm{cl}\left\{x\in X \;\middle|\; A(x)>0\right\} .$$

where $\mathrm{cl}\left\{x\in X \;\middle|\; A(x)>0\right\}$ denotes the closure of $\left\{x\in X \;\middle|\; A(x)>0\right\}$.

An α-level set of a fuzzy set $A\in\mathcal{F}(X)$ is defined by

$$A_\alpha := \begin{cases} \{t\in X \mid A(t)\geq\alpha\} & \text{if } \alpha\in(0,1] \\ \mathrm{supp}A & \text{if } \alpha=0 . \end{cases}$$

A function $T:[0,1]\times[0,1]\rightarrow[0,1]$ is said to be triangular norm (t-norm for short) iff T is symmetric, associative, non-decreasing and $T(a,1)=a$ for each $a\in[0,1]$.

Recall that if T a t-norm, $f:X\times Y\rightarrow Z$, $A\in\mathcal{F}(X)$ and $B\in\mathcal{F}(Y)$ then the fuzzy set $f(A,B)\in\mathcal{F}(Z)$ is defined via the extension principle by

$$f(A,B)(z) = \sup_{f(x,y)=z} T(A(x),B(y)), \qquad z\in Z.$$

3. The results

In this section we generalize Proposition 3.3 and Proposition 5.1 [1] to α-cuts of the extended two-place functions defined via a sup- t-norm convolution.

The following theorem illustrates that if instead of min-norm in Zadeh's extension principle we use an arbitrary t-norm, then obtain results similar to those of Nguyen.

Theorem 1. Let $X\neq\emptyset$, $Y\neq\emptyset$, $Z\neq\emptyset$ be sets and let T be a t-norm. If $f:X\times Y\rightarrow Z$ is a two-place function and $A\in\mathcal{F}(X)$, $B\in\mathcal{F}(Y)$, then a necessary and sufficient condition for the equality:

$$[f(A,B)]_\alpha = \bigcup_{T(\xi,\eta)\geq\alpha} f(A_\xi,B_\eta), \qquad \alpha\in(0,1]$$

is: for each $z\in Z$, $\sup_{f(x,y)=z} T(A(x),B(y))$ is attained.

The following theorem shows that the equality (*) holds for all u.s.c. T and continuous f in the class of u.s.c and compactly-supported fuzzy subsets.

Theorem 2. If $f:X\times Y\rightarrow Z$ is continuous and the t-norm T is u.s.c., then

$$[f(A,B)]_\alpha = \bigcup_{T(\xi,\eta)\geq\alpha} f(A_\xi,B_\eta), \qquad \alpha\in(0,1]$$

holds for each $A\in\mathcal{F}(X,\mathcal{K})$ and $B\in\mathcal{F}(Y,\mathcal{K})$.

The following examples illustrate that the α-cuts of the fuzzy set $f(A,B)$ can be generated in a simple way supposing that the t-norm in question has a simple form.

Example 1. It is easy to verify that if $T(x,y)=\min(x,y)$, then the equation (*) can be reduced to the well known form:

$$[f(A,B)]_\alpha = f(A_\alpha,B_\alpha) \qquad \alpha\in(0,1]$$

which coincides with Nguyen's result.

Example 2. If $T(x,y)=T_W(x,y)$, where

$$T_W(x,y) = \begin{cases} x & \text{if } y=1 \\ y & \text{if } x=1 \\ 0 & \text{else} \end{cases}$$

is the weak t-norm, then the equation (*) turns into

$$[f(A,B)]_\alpha = f(A_1,B_\alpha)\cup f(A_\alpha,B_1) \qquad \alpha\in(0,1],$$

hence $T_W(\xi,\eta)\geq\alpha>0$ holds only if $\xi=1$ or $\eta=1$.
Thus if $A_1=\emptyset$ or $B_1=\emptyset$, then $[f(A,B)]_\alpha=\emptyset \quad \forall\ \alpha\in(0,1]$.
If there exist unique x_0 and y_0 such that $A(x_0)=B(y_0)=1$, then we obtain

$$[f(A,B)]_\alpha = f(x_0,B_\alpha)\cup f(A_\alpha,y_0) \qquad \alpha\in(0,1].$$

Example 3. If $T(x,y)=x\cdot y$, then the equation (*) yields

$$[f(A,B)]_\alpha = \bigcup_{\xi\in[\alpha,1]} f(A_\xi,B_{\alpha/\xi}) \qquad \alpha\in(0,1].$$

Example 4. If $T(x,y)=\max\{0,x+y-1\}$, then

$$[f(A,B)]_\alpha = \bigcup_{\xi\in[\alpha,1]} f(A_\xi,B_{\alpha+1-\xi}) \qquad \alpha\in(0,1].$$

Reference

[1] Nguyen,H.T. A Note on the Extension Principle for Fuzzy Sets, *J.Math.Anal.Appl.*, **64**(1978) 369-380.