EIGEN SUBSPACE FOR FUZZY RELATION*)

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Summary. Family of all eigen fuzzy sets of a fuzzy relation is considered and characterized by relation images. In particular, a formula for the greatest eigen fuzzy set of fuzzy relation is proved.

1. Introduction. Our paper is a reflection on one result presented by Cao [1] for fuzzy relations on a finite set only. This restriction is overcomed here which gives a generalization of many results presented in [9].

Let $X \neq \emptyset$ be a fixed set and let $F(X)$ and $F(X,X)$ denote the families of fuzzy sets $A : X \to [0,1]$ and fuzzy relations $R : X \times X \to [0,1]$ on a set X. The notion of subspace of $F(X)$ was introduced in [4] and examined in [5] and [8] (for finite X).

Definition 1. Subset $E \subset F(X)$ is called a max-min subspace of $F(X)$ iff

$$A \vee B \in E, \; a \wedge A \in E \quad \text{for any} \quad A, B \in E, \; a \in [0,1].$$

Symbols $\vee$, $\wedge$, $\bigvee$, $\bigwedge$ are used for max, min, sup and inf, respectively. Later T denotes an arbitrary set of indices and also $m, n, k \in \mathbb{N}$.

2. Relation composition. Following Zadeh [7] we use the notions of sup-min composition, powers and closure of fuzzy relations $R, S \in F(X,X)$:

(1) $$(RS)(x,z) = \bigvee_{y \in X} R(x,y) \wedge S(y,z) \quad \text{for} \quad x, z \in X,$$

(2) $$R^{n+1} = R\, R^n \quad \text{for} \quad n \geq 1,$$

(3) $$R^{\vee} = \bigvee_{n \geq 1} R^n.$$

Dually we also put

(4) $$R^{\wedge} = \bigwedge_{n \geq 1} R^n.$$

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It is known that (cf. e.g. [2], § 12)

Lemma 1. Relation composition (1) is associative and infinitely distributive over summation of fuzzy relations, i.e.

(5) $$R(\bigvee_{t\in T} S_t) = \bigvee_{t\in T}(R\, S_t), \quad (\bigvee_{t\in T} R_t)S = \bigvee_{t\in T}(R_t S).$$

Additionally

(6) $$R(\bigwedge_{t\in T} S_t) \le \bigwedge_{t\in T}(R\, S_t), \quad (\bigwedge_{t\in T} R_t)S \le \bigwedge_{t\in T}(R_t S)$$

for R, S, R_t, $S_t \in F(X,X)$, $t \in T$.

Directly from (3) and (5) one gets

Corollary 1. $R^{\vee}$ commutes with R under composition (1), i.e.

(7) $$R\, R^{\vee} = R^{\vee} R.$$

The dual result with $R^{\wedge}$ will be proved later in Lemma 7. Now we consider other properties of (3) and (4).

Lemma 2. For any $R \in F(X,X)$

(8) $$R^{\vee} = R \vee R\, R^{\vee} = R \vee (R^{\vee})^2,$$

(9) $$R^{\wedge} \ge R \wedge R\, R^{\wedge}, \; R^{\wedge} \ge R \wedge (R^{\wedge})^2.$$

Proof. Directly from (3) and (5) one gets

$$(R^{\vee})^2 = R^{\vee}(\bigvee_{n\ge 1} R^n) = \bigvee_{n\ge 1} R^{\vee}(R^n) = \bigvee_{n\ge 1}\bigvee_{k\ge 1} R^{n+k} = \bigvee_{m\ge 2} R^m$$

$$= R(\bigvee_{n\ge 1} R^n) = R\, R^{\vee}.$$

Thus

$$R^{\vee} = R \vee \bigvee_{n\ge 2} R^n = R \vee R\, R^{\vee} = R \vee (R^{\vee})^2,$$

which proves (8). Similarly from (4) and (6) one obtains (9).

By mathematical induction the above proof implies

Corollary 2. For any $n \ge 1$

(10) $$(R^{\vee})^n = R^{n-1}R^{\vee} = \bigvee_{k\ge n} R^k,$$

(11) $$(R^{\wedge})^n \le \bigwedge_{k\ge n} R^k.$$

3. Relation images. Relation images for fuzzy relations were examined by Erceg [3].

Definition 2. The image of fuzzy set $A \in F(X)$ by fuzzy relation $R \in F(X,X)$ is a fuzzy set $R(A) \in F(X)$ such that

(12) $$R(A)(y) = \bigvee_{x \in X} A(x) \wedge R(x,y) \quad \text{for} \quad y \in X.$$

The family of all images of relation R is denoted by Im R, i.e.

(13) $$\mathrm{Im}\, R = \{ R(A) \mid A \in F(X) \}.$$

By analogy to Lemma 1 we can write(cf. [2], § 14)

Lemma 3. Relation image (12) is associative and infinitely distributive over summation of fuzzy sets and over summation of fuzzy relations, i.e.

(14) $$(R\, S)(A) = S(R(A)),$$

(15) $$R(\bigvee_{t \in T} A_t) = \bigvee_{t \in T} (R\, A_t), \quad (\bigvee_{t \in T} R_t)(A) = \bigvee_{t \in T} R_t(A).$$

Additionally

(16) $$R(\bigwedge_{t \in T} A_t) \le \bigwedge_{t \in T} (R\, A_t), \quad (\bigwedge_{t \in T} R_t)(A) \le \bigwedge_{t \in T} R_t(A)$$

for $R, S, R_t \in F(X,X)$, $A_t \in F(X)$, $t \in T$. In particular

(17) $$R(A \vee B) = R(A) \vee R(B), \quad (R \vee S)(A) = R(A) \vee S(A),$$

(18) $$(A \le B) \Rightarrow (R(A) \le R(B)), \quad R \le S \Rightarrow (R(A) \le S(A)),$$

(19) $$R(A \wedge B) \le R(A) \wedge R(B), \quad (R \wedge S)(A) \le R(A) \wedge S(A),$$

for $R, S \in F(X,X)$, $A, B \in F(X)$.

Moreover

(20) $$R(a \wedge A) = a \wedge R(A) \quad \text{for} \quad A \in F(X),\ a \in [0,1].$$

Now we prove another property of (12).

Lemma 4. Fuzzy relation is uniquely determined by its images. In particular $R = S$ iff

(21) $$\forall_{A \in F(X)} (R(A) = S(A)) \quad \text{for} \quad R, S \in F(X,X).$$

Proof. Any value of R can be determined by its images on "singleton " fuzzy sets

$$A_z(x) = \begin{cases} 1 & \text{for } x = z, \\ 0 & \text{otherwise.} \end{cases}$$

Really,

$$R(A_z)(y) = 0 \wedge R(x,y) \vee 1 \wedge R(z,y) = R(z,y) \quad \text{for } z,y \in X.$$

Particularly (21) implies that $R = S$.

Let us observe that properties (17) and (19) do not imply similar properties of family (13).

Example 1. Fuzzy relations on finite set can be described by suitable square matrices. Let

$$P = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.4 & 0.8 \\ 0.8 & 0.4 \end{bmatrix}, \quad R = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.2 \end{bmatrix}, \quad S = \begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}.$$

By simply verification from (13) we get (cf. Zha [8])

$\operatorname{Im} P = \{(a,a) \mid 0 \le a \le 0.2\} \cup \{(a,b) \mid 0.2 \le a \le 0.8,\ 0.2 \le b \le 0.8\}$,

$\operatorname{Im} Q = \{(a,a) \mid 0 \le a \le 0.4\} \cup \{(a,b) \mid 0.4 \le a \le 0.8,\ 0.4 \le b \le 0.8\}$,

$\operatorname{Im} R = \{(a,a) \mid 0 \le a \le 0.2\} \cup \{(a,b) \mid 0.2 \le a \le 0.4,\ 0.2 \le b \le 0.4\}$,

$\operatorname{Im} S = \operatorname{Im} Q$.

We see that $P \le Q$, $R \le P$, $R \le Q$ and Q and S are incomparable. Simultaneously $\operatorname{Im} Q \subset \operatorname{Im} P$, $\operatorname{Im} R \subset \operatorname{Im} P$, $\operatorname{Im} Q = \operatorname{Im} S$, $\operatorname{Im} R$ and $\operatorname{Im} Q$ are incomparable.

Using powers (2) we simply obtain

Theorem 1. For any $n \ge 1$

(22) $$\operatorname{Im} R^{n+1} \subset \operatorname{Im} R^n \subset \operatorname{Im} R.$$

Directly from Lemma 3 we see that

Theorem 2. $\operatorname{Im} R$ is a max-min subspace of $F(X)$ for any relation $R \in F(X,X)$.

From results of Zha [8] it can be concluded that not every max-min subspace has the form of relation image.

Example 2. The set

(23) $$E = \{(a,b) \mid 0 \le a \le 0.5,\ 0 \le b \le a+0.2\}$$

is a max-min subspace of $F(X)$ for a two-element set X. We show that the relation $R \in F(X,X)$ such that $\operatorname{Im} R = E$ does not exist.

Let us suppose that the relation

(24) $$R = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

fulfils the condition $E \subset \mathrm{Im}\, R$ with subspace (23). Using images of relation (24) for extremal points of (23) we obtain as follows:

$$p \wedge r = 0,\ p \vee r = 0.5,\ q \wedge s = 0,\ q \vee s = 0.7$$

and therefore

$$\mathrm{Im}\, R = \{(a,b) \mid 0 \le a \le 0.5,\ 0 \le b \le 0.7\} \ne E.$$

4. Eigen fuzzy sets. Eigen fuzzy sets of fuzzy relations were examined by Sanchez [6] (on a finite set).

Definition 3. $A \in F(X)$ is called an eigen fuzzy set of fuzzy relation $R \in F(X,X)$ iff $R(A) = A$. The family of all eigen fuzzy sets of given fuzzy relation R is denoted by $E(R)$, i.e.

(25) $$E(R) = \{A \in F(X) \mid R(A) = A\}.$$

We prove that

Theorem 3. $E(R)$ is a max-min subspace of $F(X)$ for any relation $R \in F(X,X)$.

Proof. Let $A, B \in E(R)$, $a \in [0,1]$. Using Lemma 3 we have

$$R(A \vee B) = R(A) \vee R(B) = A \vee B,\ R(a \wedge A) = a \wedge R(A) = a \wedge A,$$

and therefore $A \vee B \in E(R)$ and $a \wedge A \in E(R)$.

By mathematical induction one gets

Theorem 4 (cf.[9]). For any $n \ge 1$

(26) $$E(R) \subset E(R^n).$$

It is obvious that

(27) $$E(R) \subset \mathrm{Im}\, R$$

and using Theorems 1-4 we get a sequence of max-min subspaces:

$$E(R) \subset E(R^n) \subset \mathrm{Im}\, R^n \subset \ldots \subset \mathrm{Im}\, R^2 \subset \mathrm{Im}\, R$$

for any fuzzy relation R. Some sets in this sequence can coincide, which is the matter of further examination. But the first significant result concerns the closure (3).

Theorem 5. For any fuzzy relation

(28) $$E(R^{\vee}) = E(R).$$

Proof. Obviously $E(R) \subset E(R^{\vee})$ because of (3) and (25) (cf. also [9]). Assume that $A \in E(R^{\vee})$, i.e. $R^{\vee}(A) = A$. Using Lemma 1 we see that

$$R(A) = R(R^{\vee}(A)) = R(\bigvee_{n\geq 1} R^n) = \bigvee_{n\geq 2} R^n,$$

$$A = R^{\vee}(A) = \bigvee_{n\geq 1} R^n(A) = R(A) \vee \bigvee_{n\geq 2} R^n(A) = R(A) \vee R(A) = R(A)$$

and it is shown that $A \in E(R)$. Therefore $E(R^{\vee}) \subset E(R)$, which finishes the proof of property (28).

We shall examine further properties of $E(R)$ for diverse classes of fuzzy relations.

5. Idempotent relations. A fuzzy relation R is called idempotent iff

(29) $\quad R^2 = R.$

Then $R^n = R = R^{\vee}$ and obviously $E(R^n) = E(R)$. Furthermore, we have

Theorem 6. Fuzzy relation R is idempotent iff

(30) $\quad E(R) = \operatorname{Im} R.$

Proof. In virtue of (27) property (30) is equivalent to

$$R^2(A) = R(R(A)) = R(A) \quad \text{for} \quad A \in F(X).$$

By Lemma 4 this relation is equivalent to (29), which finishes the proof.

Now we ask for idempotent $R^{\vee}$. Lemma 2 directly implies (cf. property (8))

Lemma 5. Closure (3) is idempotent iff

(31) $\quad R \leq R\, R^{\vee}.$

As a simple consequence we get

Corollary 3. If

(32) $\quad R \leq R^2,$

then closure of R is idempotent.

Using Theorem 5 with Lemma 5 in Theorem 6 one obtains

Theorem 7. For any fuzzy relation

(33) $\quad E(R) = \operatorname{Im} R^{\vee},$

iff R fulfils (31). In particular (32) implies (33).

6. Transitive relations. Now we consider condition converse to inequality (32).

Definition 4 (cf. Zadeh [7]). Fuzzy relation R is called transitive iff

(34) $R^2 \le R.$

It is obvious that

Lemma 6. For any transitive relation R,

(35) $\bigvee_{n\ge 1} R^{n+1} \le R^n \le R,$

(36) $R^\wedge = \lim_{n\to\infty} R^n.$

Similarly as in [9] one can obtain

Theorem 8. For any transitive relation R,

(37) $E(R^n) = E(R).$

Now we prove

Lemma 7. Any transitive relation R commutes with $R^\wedge$. Moreover

(38) $R\,R^\wedge = R^\wedge R = R^\wedge.$

Proof. Assume (34) and let $x, z \in X$. Directly from (1) and (4), by continuity of operation sup we get

$$(R\,R^\wedge)(x,z) = \bigvee_{y\in X} R(x,y) \wedge R^\wedge(y,z) = \bigvee_{y\in X} \lim_{n\to\infty} R^n(y,z)$$

$$= \lim_{n\to\infty} (\bigvee_{y\in X} R(x,y) \wedge R^n(y,z)) = \lim_{n\to\infty} R^{n+1}(x,z) = R^\wedge(x,z)$$

and similarly

$$R^\wedge R = R^\wedge.$$

Theorem 9. For any transitive relation R,

(39) $E(R) = \mathrm{Im}\, R^\wedge.$

Proof. Obviously

$$E(R) \subset \mathrm{Im}\, R^\wedge,$$

because $R(A) = A$ implies $R^\wedge(A) = A$. Conversely let $A \in \mathrm{Im}\, R^\wedge$, i.e. $A = R^\wedge(B)$ for certain $B \in F(X)$. Then by Lemma 7

$$R(A) = R(R^\wedge(A)) = (R^\wedge R(B)) = R^\wedge(B) = A,$$

and therefore $A \in E(R)$, which proves the converse inclusion.

7. Main result. It is known that the closure (3) is transitive (compare (8) and (34)). Thus $E(R^{\vee})$ can be described by Theorem 9. But by Theorem 5 the same description has $E(R)$. Therefore

Theorem 10. For any fuzzy relation R

(40) $$E(R) = \mathrm{Im}\,(R^{\vee})^{\wedge}.$$

It is obvious that (30), (33) and (39) are special cases of (40) under respective assumption. Furthermore the formula presented by Cao [1] is a special case of (40) for card X = n.

Parallely, there are many algorithms for construction of the greatest eigen fuzzy set of fuzzy relation (cf. [6] and [9]). Now formula (40) provides direct solution of this problem, because it suffices to calculate the greatest image of suitable fuzzy relation. Therefore we get

Theorem 11. The greatest eigen fuzzy set of fuzzy relation R is described by

$$A(z) = \bigvee_{x \in X} (R^{\vee})^{\wedge}(x,z).$$

References

[1] Z.G.Cao, Eigenvectors of fuzzy matrices under incline operations, Fuzzy Math. 6 (4) (1986), 53-56.

[2] J.Drewniak, Podstawy teorii zbiorów rozmytych, Uniwersytet Śląski, Katowice 1984.

[3] M.A.Erceg, Functions, equivalence relations, quotient spaces and subsets in fuzzy set theory, Fuzzy Sets Syst.3 (1980), 75-92.

[4] K.H.Kim, F.W.Rousch, Generalized fuzzy matrices, Fuzzy Sets Syst.4 (1980), 293-315.

[5] S.H.Li, The equations of some geometric patterns, BUSEFAL 16 (1983), 38-43.

[6] E.Sanchez, Eigen fuzzy sets and fuzzy relations, J.Math.Anal.Appl.81 (1981), 399-421.

[7] L.A.Zadeh, Similarity relations and fuzzy orderings, Inform.Sci.3 (1971), 177-200.

[8] J.L.Zha, The dependence of a system of fuzzy vectors and the rank of a fuzzy matrix, BUSEFAL 20 (1984), 98-105.

[9] J.S.Zhu, R.Z.Cheng, Some properties of fuzzy eigen set, BUSEFAL 30 (1987), 51-57.