Biimplication Operator and Its Application on Fuzzy Relation Equation

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Abstract: In this paper, a biimplication operator "$\leftrightarrow$" is defined on totally ordered complete lattice L. On L and the set of L-fuzzy matrices, we discuss some basic properties of operator "$\leftrightarrow$", obtain some solutions of fuzzy relation equation $A \cdot X = A$ ($X \cdot A = A$) and fuzzy relatoin unequal equation $A \cdot X \leqq A$ ($X \cdot A \leqq A$) respectively. A sufficient and necessary condition of a L-fuzzy matrix having generalized subinverse and the generalized inverses of idempotent matrices are given.

Keywords: Biimplication operator, Transitive matrix, Idempotent matrix, Fuzzy relation equation.

1. Preparation

[1, 2] show that operator "α" (i.e. "$\rightarrow$") is important on studying fuzzy relation equation and generalized subinverse. In this paper, a biimplication operator "$\leftrightarrow$" is defined on lattice L, and we will discuss some basic properties of the operator "$\leftrightarrow$" on lattice L and the set of L-fuzzy matrices. In the following, let L be a totally ordered complete lattice, 0, 1 denote the least element and greatest element of L respectively. $L^{n\times m}$ denotes the set of all of the nxm fuzzy matrices, A^T is the transposed matrix of A, I_n is n-order identity matrix. In this paper, we need some concepts of the following.

Definition 1.1. Let $a, b \in L$, define

$$a \to b = \begin{cases} 1 & \text{if} \quad a \leqq b \\ b & \text{if} \quad a > b \end{cases}$$

and $a \leftarrow b = b \to a$.

Definition 1.2. Let $A=(a_{ij}) \in L^{n\times m}$, $B=(b_{ij}) \in L^{m\times n}$, define $A \cdot B = (\bigvee_{k=1}^{m}(a_{ik} \wedge b_{kj}))_{n\times n}$. And for any $C \in L^{n\times n}$, let $C^0 = I_n$, $C^1 = C$, $C^2 = C^1 \cdot C, \cdots$, $C^m = C^{m-1} \cdot C$. If there exists a popitive interger k, such that $C^k = C^{k+1}$, then we say C is convergence of power.

Definition 1.3. Let $A \in L^{n\times m}$, if there exists $B \in L^{m\times n}$, such that $A \cdot B \cdot A = A$, then A is called regular, in this case, B is called a generalized inverse of A. If $A \cdot B \cdot A \leqq A$, then B is called a generalized subinverse of A.

Definition 1.4. Let $B \in L^{n\times n}$ be an L-fuzzy symmetric square matrix, if there exists $A \in L^{n\times m}$, such that $B = A \cdot A^T$, then B is called realizable.

Definition 1.5. Let $A=(a_{ij}) \in L^{n\times n}$, if $a_{ii} \geqq a_{ik} \vee a_{ki}$, $1 \leqq i$, $k \leqq n$, then A is called a diagonally dominant matrix.

2. Biimplication Operator "$\leftrightarrow$"

Definition 2.1. Let $a, b \in L$, define $a \leftrightarrow b = (a \leftarrow b) \wedge (a \to b)$, that is,

$$a \leftrightarrow b = \begin{cases} b & \text{if} \quad a > b \\ 1 & \text{if} \quad a = b \\ a & \text{if} \quad a < b. \end{cases}$$

Easily to see $a \leftrightarrow b = b \leftrightarrow a$

Proposition 2.1. Let $a, b, c \in L$, then $(a \leftrightarrow c) \wedge (b \leftrightarrow c) \leqq a \leftrightarrow b$

Proof. (1) If $c \neq a, b$, then $(a \leftrightarrow c) \wedge (b \leftrightarrow c) = a \wedge b \wedge c \leqq a \wedge b \leqq a \leftrightarrow b$.

(2) If $c = a$, but $c \neq b$, then $(a \leftrightarrow c) \wedge (b \leftrightarrow c) = b \leftrightarrow c = b \leftrightarrow a = a \leftrightarrow b$.

(3) If $c \neq a$, but $c = b$, similarly with (2), the conclusion holds.

(4) If $a = b = c$, the conclusion holds obviously.

Proposition 2.2. Let $a, b \in L$, then $a \wedge (a \leftrightarrow b) \leqq b$.

Proof. Since

$$a \wedge (a \leftrightarrow b) = \begin{cases} b & \text{if} \quad a > b \\ a & \text{if} \quad a = b \\ a & \text{if} \quad a < b, \end{cases}$$

then $a\wedge(a\leftrightarrow b)\leq b$.

Proposition 2.3. Let $a, b, c\in L$, then

(i) $(a\wedge b)\leftrightarrow c=[(b\leftarrow c)\wedge(a\leftrightarrow c)]\vee[(a\leftarrow c)\wedge(b\leftrightarrow c)]$;

(ii) $(a\vee b)\leftrightarrow c=[(a\leftrightarrow c)\wedge(b\rightarrow c)]\vee[(a\rightarrow c)\wedge(b\leftrightarrow c)]$.

Proof. (i) Easily to see $(a\wedge b)\leftarrow c=(a\leftarrow c)\wedge(b\leftarrow c)$, $(a\wedge b)\rightarrow c=(a\rightarrow c)\vee(b\rightarrow c)$.

$$
\begin{aligned}
(a\wedge b)\leftrightarrow c&=[(a\wedge b)\leftarrow c]\wedge[(a\wedge b)\rightarrow c]\\
&=[(a\leftarrow c)\wedge(b\leftarrow c)]\wedge[(a\rightarrow c)\vee(b\rightarrow c)]\\
&=(a\leftarrow c)\wedge\{[(b\leftarrow c)\wedge(a\rightarrow c)]\vee[(b\leftarrow c)\wedge(b\rightarrow c)]\}\\
&=(a\leftarrow c)\wedge\{[(b\leftarrow c)\wedge(a\rightarrow c)]\vee(b\leftrightarrow c)\}\\
&=[(a\leftarrow c)\wedge(b\leftarrow c)\wedge(a\rightarrow c)]\vee[(a\leftarrow c)\wedge(b\leftrightarrow c)]\\
&=[(a\leftrightarrow c)\wedge(b\leftarrow c)]\vee[(a\leftarrow c)\wedge(b\leftrightarrow c)]\\
&=[(b\leftarrow c)\wedge(a\leftrightarrow c)]\vee[(a\leftarrow c)\wedge(b\leftrightarrow c)].
\end{aligned}
$$

Similarly, we can verify (ii).

Proposition 2.4. Let $a, b, c\in L$, then

(i) $(a\leftrightarrow c)\wedge(b\leftrightarrow c)\leq(a\wedge b)\leftrightarrow c\leq(a\leftrightarrow c)\vee(b\leftrightarrow c)$;

(ii) $(a\leftrightarrow c)\wedge(b\leftrightarrow c)\leq(a\vee b)\leftrightarrow c\leq(a\leftrightarrow c)\vee(b\leftrightarrow c)$.

Proof. In the following, we only examine (i), (ii) can be proved similarly.

(1) If $a>b$, then

$$(a\leftrightarrow c)\wedge(b\leftrightarrow c)=\begin{cases}(a\leftrightarrow c)\wedge c & \text{if } b>c\\ c & \text{if } b=c\\ (a\leftrightarrow c)\wedge b & \text{if } b<c,\end{cases}\qquad (a\wedge b)\leftrightarrow c=b\leftrightarrow c=\begin{cases}c & \text{if } b>c\\ 1 & \text{if } b=c\\ b & \text{if } b<c.\end{cases}$$

(2) If $a=b$, then

$$(a\leftrightarrow c)\wedge(b\leftrightarrow c)=\begin{cases}c & \text{if } b>c\\ 1 & \text{if } b=c\\ b & \text{if } b<c\end{cases}$$
$$=(a\wedge b)\leftrightarrow c$$

(3) If $a<b$, then

$$(a\leftrightarrow c)\wedge(b\leftrightarrow c)=\begin{cases}(b\leftrightarrow c)\wedge c & \text{if } a>c\\ c & \text{if } a=c\\ a\wedge(b\leftrightarrow c) & \text{if } a<c,\end{cases}\qquad (a\wedge b)\leftrightarrow c=\begin{cases}c & \text{if } a>c\\ 1 & \text{if } a=c\\ a & \text{if } a<c.\end{cases}$$

Through compared with, we can see for any $a, b, c\in L$, $(a\leftrightarrow c)\wedge(b\leftrightarrow c)\leq(a\wedge b)\leftrightarrow c$. By Proposition 2.3 (i), $(a\wedge b)\leftrightarrow c=[(b\leftarrow c)\wedge(a\leftrightarrow c)]\vee[(a\leftarrow c)\wedge(b\leftrightarrow c)]\leq(a\leftrightarrow c)\vee(b\leftrightarrow c)$.

Therefore, $(a\leftrightarrow c)\wedge(b\leftrightarrow c)\leqq(a\wedge b)\leftrightarrow c\leqq(a\leftrightarrow c)\vee(b\leftrightarrow c)$.

From Proposition 2.3, 2.4, we can see, generally, for any a, b, $c\in L$, $(a\wedge b)\leftrightarrow c=(a\leftrightarrow c)\wedge(b\leftrightarrow c)$ and $(a\vee b)\leftrightarrow c=(a\leftrightarrow c)\vee(b\leftrightarrow c)$ are not satisfied. But the following propositions show, under some conditions, they hold yet.

Theorem 2.5. Let a, b, $c\in L$, if $a\wedge b\neq c$, then

(i) $(a\wedge b)\leftrightarrow c=(a\leftrightarrow c)\wedge(b\leftrightarrow c)$;

(ii) $(a\vee b)\leftrightarrow c=(a\leftrightarrow c)\vee(b\leftrightarrow c)$.

Proof. (i) If $a\wedge b>c$, then $(a\wedge b)\leftrightarrow c=c$, while $(a\leftrightarrow c)\wedge(b\leftrightarrow c)=c\wedge c=c$. If $a\wedge b<c$, then $a<c$ or $b<c$. Suppose $a<c$, thus

$$(a\wedge b)\leftrightarrow c=(a\wedge b)=\begin{cases} a & \text{if } b>c \\ a & \text{if } b=c \\ a\wedge b & \text{if } b<c, \end{cases}$$

$$(a\leftrightarrow c)\wedge(b\leftrightarrow c)=a\wedge(b\leftrightarrow c)=\begin{cases} a & \text{if } b>c \\ a & \text{if } b=c \\ a\wedge b & \text{if } b<c, \end{cases}$$

hence $(a\wedge b)\leftrightarrow c=(a\leftrightarrow c)\wedge(b\leftrightarrow c)$

(ii) Since $a\wedge b\neq c$, then suppose $a\neq c$, if $b=c$, then $a<c=b$, in this case, $(a\vee b)\leftrightarrow c=b\leftrightarrow c=1$, $(a\leftrightarrow c)\vee(b\leftrightarrow c)=(a\leftrightarrow c)\vee 1=1$, hence (ii) holds. If $a=b$, then, obvious, $(a\vee b)\leftrightarrow c=(a\leftrightarrow c)\vee(b\leftrightarrow c)$. By the way, we only need to prove, when $a\neq b$, $b\neq c$, $c\neq a$, (ii) holds, too, then (ii) has been proved. In fact, when $a\neq b$, $b\neq c$, $c\neq a$, there only exsit six cases among a, b, c, i.e. $a>b>c$, $a>c>b$, $b>c>a$, $b>a>c$, $c>a>b$, $c>b>a$. For any cases, we can examine (ii) holds.

Proposition 2.6. Let a, b, $c\in L$, and $c\neq 1$, then $(a\wedge b)\leftrightarrow c=(a\leftrightarrow c)\wedge(b\leftrightarrow c)$. if and only if one of the following condition holds.

(i) when $a>b$, $b\neq c$;

(ii) when $a<b$, $a\neq c$;

(iii) when $a=b$.

Proof. If $a\neq b$, then when $a>b$, certainly $b\neq c$, otherwise, $(a\wedge b)\leftrightarrow c=b\leftrightarrow c=1$, $(a\leftrightarrow c)\wedge(b\leftrightarrow c)=c\wedge 1=c$, but $c\neq 1$, contradiction.

Similarly, when $a<b$, we have $a \neq c$. Conversely, (1) when $a=b$, the conclusion is obvious. (2) when $a>b$, since $b \neq c$, hence $a \wedge b=b \neq c$; when $a<b$, since $a \neq c$, hence $a \wedge b=a \neq c$, in a word, $a \wedge b \neq c$, by Theorem 2.5 (i), we can verify the conclusion.

Proposition 2.7. Let a, b, $c \in L$, and $c \neq 1$, then $(a \vee b) \leftrightarrow c=(a \leftrightarrow c) \vee (b \leftrightarrow c)$. if and only if one of the following conditions holds.

(i) when $a>b$, $b \neq c$;

(ii) when $a<b$, $a \neq c$;

(iii) when $a=b$.

Proof. If $a \neq b$, then when $a>b$, certainly $b \neq c$, otherwise, $(a \vee b) \leftrightarrow c=a \leftrightarrow c=c$, $(a \leftrightarrow c) \vee (b \leftrightarrow c)=c \vee 1=1$, but $c \neq 1$, contradiction. Similarly, when $a<b$, we have $a \neq c$. Conversely, (1) when $a=b$, then the conclusion is obvious. (2) when $a>b$, since $b \neq c$, thus $a \wedge b=b \neq c$; when $a<b$, since $a \neq c$, hence $a \wedge b=a \neq c$, in a word, $a \wedge b \neq c$, by Theorem 2.5(ii), the conclusion holds.

Now we discuss the biimplication operator on L-fuzzy matrices.

Definition 2.2. Let $A=(a_{ij}) \in L^{n \times m}$, $B=(b_{ij}) \in L^{m \times n}$, define $A \leftarrow B=(\bigwedge_{k=1}^{m}(a_{ik} \leftarrow b_{kj}))_{n \times n}$, $A \rightarrow B=(\bigwedge_{k=1}^{m}(a_{ik} \rightarrow b_{kj}))_{n \times n}$, $A \leftrightarrow B=(A \leftarrow B) \wedge (A \rightarrow B)$.

According to the definition, $A \leftrightarrow B=(\bigwedge_{k=1}^{m}(a_{ik} \leftrightarrow b_{kj}))_{n \times n}$, Remark, generally $A \leftrightarrow B \neq B \leftrightarrow A$. For example, let $L=[0, 1]$,

$$A=\begin{pmatrix} 0.2 & 0.3 \\ 0.4 & 0.5 \end{pmatrix}, \qquad B=\begin{pmatrix} 0.7 & 0.8 \\ 0.1 & 0.6 \end{pmatrix},$$

then we have

$$A \leftrightarrow B=\begin{pmatrix} 0.1 & 0.2 \\ 0.1 & 0.4 \end{pmatrix}, \text{ but } \qquad B \leftrightarrow A=\begin{pmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{pmatrix}.$$

Proposition 2.8. For any $A \in L^{n \times m}$, then (1) $A \rightarrow A^T \geq I_n$; (2) $A \leftarrow A^T \geq I_n$; (3) $(A \rightarrow A^T) \cdot A \geq A$.

Proof. (1) Let $A=(a_{ij})_{n \times m}$, thus $A \rightarrow A^T=(\bigwedge_{k=1}^{m}(a_{ik} \rightarrow a_{jk}))_{n \times n}$, since $\bigwedge_{k=1}^{m}(a_{ik} \rightarrow a_{ik})=1$, $1 \leq i \leq n$, hence $A \rightarrow A^T \geq I_n$; (2) see [2]; (3) can be verified at once by right multiplying A on two sides of (1).

Theorem 2.9. Let $A \in L^{n\times m}$, then

(1) $A \leftrightarrow A^T$ is a reflexive matrix (i.e. $A \leftrightarrow A^T \geqq I_n$);

(2) $A \leftrightarrow A^T$ is a symmetric matrix;

(3) $A \leftrightarrow A^T$ is a idempotent matrix;

(4) $A \leftrightarrow A^T$ is a power convergent matrix;

(5) $A \leftrightarrow A^T$ is a diagonally dominant matrix;

(6) $A \leftrightarrow A^T$ is realizable.

Proof. (1) From Proposition 2.8, $A \leftrightarrow A^T = (A \leftarrow A^T) \wedge (A \rightarrow A^T) \geqq I_n \wedge I_n = I_n$, i.e. $A \leftrightarrow A^T \geqq I_n$.

(2) $A \leftrightarrow A^T = (a_{ij})_{n\times m} \leftrightarrow (a_{ij})^T_{n\times m} = (\bigwedge_{k=1}^{m} (a_{ik} \leftrightarrow a_{jk}))_{n\times n}$,

let $A \leftrightarrow A^T = (r_{ij})_{n\times n}$, then $r_{ij} = \bigwedge_{k=1}^{m} (a_{ik} \leftrightarrow a_{jk}) = \bigwedge_{k=1}^{m} (a_{jk} \leftrightarrow a_{ik}) = r_{ji}$, $1 \leqq i, j \leqq n$, hence $A \leftrightarrow A^T = (A \leftrightarrow A^T)^T$.

(3) $$(A \leftrightarrow A^T) \cdot (A \leftrightarrow A^T) = (\bigwedge_{k=1}^{m} (a_{ik} \leftrightarrow a_{jk}))_{n\times n} \cdot (\bigwedge_{k=1}^{m} (a_{ik} \leftrightarrow a_{jk}))_{n\times n}$$
$$= (\bigvee_{f=1}^{n} [(\bigwedge_{k=1}^{m} (a_{ik} \leftrightarrow a_{fk})) \wedge (\bigwedge_{k=1}^{m} (a_{fk} \leftrightarrow a_{jk}))])_{n\times n}$$
$$= (\bigvee_{f=1}^{n} [\bigwedge_{k=1}^{m} ((a_{ik} \leftrightarrow a_{fk}) \wedge (a_{jk} \leftrightarrow a_{fk}))])_{n\times n}$$

By Proposition 2.1.

$$(a_{ik} \leftrightarrow a_{fk}) \wedge (a_{jk} \leftrightarrow a_{fk}) \leqq a_{ik} \leftrightarrow a_{jk}, \quad 1 \leqq k \leqq m, \quad 1 \leqq i,j,f \leqq n.$$

Therefore

$$(A \leftrightarrow A^T) \cdot (A \leftrightarrow A^T) \leqq (\bigvee_{f=1}^{n} [\bigwedge_{k=1}^{m} (a_{ik} \leftrightarrow a_{jk})])_{n\times n}$$
$$= (\bigwedge_{k=1}^{m} (a_{ik} \leftrightarrow a_{jk}))_{n\times n} = A \leftrightarrow A^T$$

On the other hand

$$(A \leftrightarrow A^T) \cdot (A \leftrightarrow A^T) = (\bigvee_{f=1}^{n} [\bigwedge_{k=1}^{m} ((a_{ik} \leftrightarrow a_{fk}) \wedge (a_{jk} \leftrightarrow a_{fk}))])_{n\times n}$$
$$\geqq (\bigwedge_{k=1}^{m} ((a_{ik} \leftrightarrow a_{jk}) \wedge (a_{jk} \leftrightarrow a_{jk})))_{n\times n}$$
$$= (\bigwedge_{k=1}^{m} (a_{ik} \leftrightarrow a_{jk}))_{n\times n} = A \leftrightarrow A^T$$

Consequently $(A \leftrightarrow A^T)^2 = A \leftrightarrow A^T$.

(4) can be got at once by (3).

(5) can be obtained at once by (1).

(6) can be verified immediately by (2), (3).

Remark. In general, for any $A \in L^{n\times n}$, $A \leftrightarrow A \neq A$. For example, let $L = [0, 1]$.

$$A = \begin{pmatrix} 0.3 & 0.1 & 0.7 \\ 0.8 & 0.2 & 0.6 \\ 0.6 & 0.5 & 0.4 \end{pmatrix},$$

then

$$A \leftrightarrow A = \begin{pmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.2 \\ 0.3 & 0.1 & 0.5 \end{pmatrix},$$

easily to see $A \leftrightarrow A \neq A$. But we have conclusions of the following:

Proposition 2.10. If $A=(a_{ij}) \in L^{n\times n}$ is a weakly reflexive matrix (i.e. for $1 \leq i, j \leq n$, $a_{ij} \leq a_{ii}$), for $i \neq f$ or $j \neq k$, $a_{ij} \neq a_{fk}$, then $A \leftrightarrow A \leq A$.

Proof. Since $A \leftrightarrow A = (\bigwedge_{k=1}^{n}(a_{ik} \leftrightarrow a_{kj}))_{n\times n}$, hence $\bigwedge_{k=1}^{n}(a_{ik} \leftrightarrow a_{kj}) \leq a_{ii} \leftrightarrow a_{ij} \leq a_{ii} \wedge a_{ij} = a_{ij}$, $1 \leq i,j \leq n$. Therefore $A \leftrightarrow A \leq A$.

Proposition 2.11. For any $A \in L^{n\times n}$, if A is a reflexive matrix, then

(1) $(A^T \leftrightarrow A) \leq A$;

(2) $(A \leftrightarrow A^T) \leq A$.

Proof. (1) Since $A \geq I_n$, hence $a_{ii}=1$, $1 \leq i \leq n$. As a result, $A^T \leftrightarrow A = (\bigwedge_{k=1}^{n}(a_{ki} \leftrightarrow a_{kj}))_{n\times n} \leq ((a_{ii} \leftrightarrow a_{ij}))_{n\times n} = (a_{ij})_{n\times n} = A$, i.e. $A^T \leftrightarrow A \leq A$.

(2) can be verified by the same reason.

Proposition 2.12. For any $A \in L^{n\times m}$, if $a_{ij} \neq 0$, $1 \leq i \leq n$, $1 \leq j \leq m$, then $A \leftrightarrow I_m = 0_{n\times m}$ (the nxm zero matrix).

Proof. Since $A \leftrightarrow I_m = (\bigwedge_{k=1}^{m}(a_{ik} \leftrightarrow \delta_{kj}))_{n\times m}$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\bigwedge_{k=1}^{m}(a_{ik} \leftrightarrow \delta_{kj}) = (a_{i1} \leftrightarrow \delta_{1j}) \wedge \cdots \wedge (a_{ij} \leftrightarrow \delta_{jj}) \wedge \cdots \wedge (a_{im} \leftrightarrow \delta_{mj}) = 0 \wedge 0 \wedge \cdots \wedge 0 \wedge (a_{ij} \leftrightarrow \delta_{jj}) \wedge 0 \wedge \cdots \wedge 0 = 0$, $1 \leq i \leq n$, $1 \leq j \leq m$. Hence $A \leftrightarrow I_m = 0_{n\times m}$.

Proposition 2.13. For any $A \in L^{n\times m}$, then

(1) $(A \leftrightarrow A^T) \cdot A = A$;

(2) $A \cdot (A^T \leftrightarrow A) = A$.

Proof. We prove (1) only, (2) can be verified similarly. Let $(A \leftrightarrow A^T) \cdot A = (r_{ij})_{n\times m}$, then

$$(r_{ij})_{n\times m} = (\bigwedge_{k=1}^{m}(a_{ik} \leftrightarrow a_{jk}))_{n\times n} \cdot (a_{ij})_{n\times m}$$

$$=\left(\bigvee_{f=1}^{n}\left[\bigwedge_{k=1}^{m}((a_{ik}\leftrightarrow a_{fk})\wedge a_{fj})\right]\right)_{n\times m}$$

thus

$$r_{ij}=\bigvee_{f=1}^{n}\left[\bigwedge_{k=1}^{m}((a_{ik}\leftrightarrow a_{fk})\wedge a_{fj})\right]$$
$$=\left[\bigwedge_{k=1}^{m}((a_{ik}\leftrightarrow a_{1k})\wedge a_{1j})\right]\vee\cdots\vee\left[\bigwedge_{k=1}^{m}((a_{ik}\leftrightarrow a_{ik})\wedge a_{ij})\right]$$
$$\vee\cdots\vee\left[\bigwedge_{k=1}^{m}((a_{ik}\leftrightarrow a_{nk})\wedge a_{nj})\right],\qquad 1\leq i\leq n,\ 1\leq j\leq m.$$

Once more, by Proposition 2.2, for any $h\neq i$,

$$\bigwedge_{k=1}^{m}((a_{ik}\leftrightarrow a_{hk})\wedge a_{hj})\leq(a_{ij}\leftrightarrow a_{hj})\wedge a_{hj}\leq a_{ij},\qquad 1\leq i,\ h\leq n,\ 1\leq j\leq m.$$

Yet $\bigwedge_{k=1}^{m}((a_{ik}\leftrightarrow a_{ik})\wedge a_{ij})=\bigwedge_{k=1}^{m}a_{ij}=a_{ij}$, $1\leq i\leq n$, $1\leq j\leq m$. Therefore $(A\leftrightarrow A^T)\cdot A=A$.

3. Application on Fuzzy Relation Equation

In this section, the solutions of fuzzy relation unequal equation and fuzzy relation equation are discussed by using properties of biimplication operator, we obtain some results. In following, suppose X is a fuzzy unknown square matrix such that $A\cdot X(X\cdot A)$ having meaning.

By Proposition 2.8 (3), the following proposition holds.

Proposition 3.1. For any $A\in L^{n\times m}$, $X\cdot A\geq A$ always has a solution $A\rightarrow A^T$.

Proposition 3.2. If $A=(a_{ij})\in L^{n\times n}$ is a idempotent matrix and for $i\neq f$ or $j\neq k$, $a_{ij}\neq a_{fk}$, $1\leq i, j, f, k\leq n$. Then $A\cdot X\leq A(X\cdot A\leq A)$ always has a solution $A\leftrightarrow A$.

Proof. Since $A^2=A$, then $\left(\bigvee_{k=1}^{n}(a_{ik}\wedge a_{kj})\right)_{n\times n}=(a_{ij})_{n\times n}$, hence

$$a_{ij}=\bigvee_{k=1}^{n}(a_{ik}\wedge a_{kj})=(a_{i1}\wedge a_{1j})\vee\cdots\vee(a_{ii}\wedge a_{ij})\vee\cdots\vee(a_{in}\wedge a_{nj}).$$

Since for $i\neq f$ or $j\neq k$, $a_{ij}\neq a_{fk}$, $1\leq i,j,f,k\leq n$.

Therefore, certainly

$$a_{ij}=(a_{i1}\wedge a_{1j})\vee\cdots\vee(a_{ii}\wedge a_{ij})\vee\cdots\vee(a_{in}\wedge a_{nj})=a_{ii}\wedge a_{ij}\leq a_{ii},$$

$1\leq i, j\leq n$, that is, A is a weakly reflexive matrix. By Proposition 2.10, $A\leftrightarrow A\leq A$, hence $A\cdot(A\leftrightarrow A)\leq A\cdot A=A^2=A$. Consequently, $A\cdot X\leq A$, always has a solution $A\leftrightarrow A$. By the same reason, we can prove $X\cdot A\leq A$ always has a solution $A\leftrightarrow A$ also.

Lemma 3.3. Let $B=(b_{ij})\in L^{n\times n}$, $n\geqq 2$, and $b_{ij}\leqq b_{ii}$, $1\leqq i, j\leqq n$, then $B\leqq B^2\leqq\cdots\leqq B^{n-1}=B^n=\cdots$

The proof can be found in [3].

Corollary 3.4. For any $A=(a_{ij})\in L^{n\times n}$, if A satisfies for $i\neq f$ or $j\neq k$, $a_{ij}\neq a_{fk}$, $1\leqq i, j, f, k\leqq n$, then a sufficient and necessary condition that A is a idempotent matrix is that A is a weakly reflexive matrix and $A^2\leqq A$.

Proof. From the proving process of Proposition 3.2, we know if $A^2=A$, then A is a weakly reflexive matrix, $A^2\leqq A$ naturally. Conversely, since A is a weakly reflexive matrix, by Lemma 3.3, $A\leqq A^2$. Yet $A^2\leqq A$, hence $A^2=A$.

Proposition 3.5. For any $A\in L^{n\times m}$, $A\cdot X=A$, $X\cdot A=A$ has solution $A^T\leftrightarrow A$, $A\leftrightarrow A^T$, respectively.

Proof. This can be examined directly by Proposition 2.13.

Proposition 3.6. For any $A\in L^{n\times n}$, then the following conditions are equivalent:

(1) A is a transitive matrix;

(2) $A\cdot(A\leftrightarrow A^T)\cdot A\leqq A$;

(3) $A\cdot(A^T\leftrightarrow A)\cdot A\leqq A$.

Proof. We only verify (1)$\Leftrightarrow$(2).

(1)$\Rightarrow$(2). If $A^2\leqq A$, by Proposition 2.13, $A\cdot(A\leftrightarrow A^T)\cdot A=A\cdot[(A\leftrightarrow A^T)\cdot A]=A\cdot A=A^2\leqq A$, that is, (2) holds.

(2)$\Rightarrow$(1). If (2) holds, i.e. $A\cdot(A\leftrightarrow A^T)\cdot A\leqq A$, yet $(A\leftrightarrow A^T)\cdot A=A$, hence $A\cdot A\leqq A$, that is, (1) holds.

Proposition 3.7. For any $A\in L^{n\times n}$, if $A^2=A$, then both $A\leftrightarrow A^T$ and $A^T\leftrightarrow A$ are generalized inverse of A.

Proof. By Proposition 2.13, $(A\leftrightarrow A^T)\cdot A=A$, $A\cdot(A^T\leftrightarrow A)=A$, hence $A\cdot(A\leftrightarrow A^T)\cdot A=A^2=A$, $A\cdot(A^T\leftrightarrow A)\cdot A=A\cdot A=A^2=A$, that is, $A\cdot(A\leftrightarrow A^T)\cdot A=A$, $A\cdot(A^T\leftrightarrow A)\cdot A=A$.

References

[1] E. Sanchez, Solution of Fuzzy Equation with Extended Operations. Fuzzy Sets and Systems, 12(1984), 237-248.

[2] H. Hashimoto, Subinverse of Fuzzy Matrices, Fuzzy Sets and Systems, 12(1984), 155-168.

[3] Liu Wang-jin, The Realizable Problem for Fuzzy Symmetric Matrix. Fuzzy Mathematics (Chinese), 1(1982), 69-75.

[4] K. H. Kim and F. W. Roush. Generalized Fuzzy Matrices, Fuzzy Sets and Systems, 4(1980), 293-316.