The Discrete Expression of Fundamental Theorems In Fuzzy set Theory

Wang Shu-ze and Liu Xi-qiang
Dept. of Math., Liaocheng Teachers' College,
Shandong, People's Republic of China

ABSTRACT

In this paper, the discrete forms of the three fundamental theorems: decomposition, expression and extension theorem in fuzzy set theory are discussed. We have proved that a fuzzy set can be expressed by a sequence of ordinary sets and approximated by the finite ordinary sets.Perhaps,it is benefit to practial application.

§1 Introduction

In fuzzy set theory, the decomposition, expression and extension theorems are three fundamental theorems. and they play an important role. It is well-known that these theorems describe the relation between ordinary set and fuzzy set. For example, a fuzzy set A can be denoted by

$$A = \bigcup_{\lambda\in[0,1]} \lambda H(\lambda) \qquad (1.1)$$

Where $\{H(\lambda) \mid \lambda\in[0,1]\}$ is a family of sets with some properties. In(1.1), even if the set $H(\lambda)$ for every $\lambda\in[0.1]$ could be known, it is difficult to use them to find A or the membership function A(x). The reason is that λ changes in the interval [0,1] which is continuous cardinal number, and the value of A(x) is obtained by taking sup for the uncountable values. In this paper, the interval [0,1] will be replaced by its countable dense subset in (1.1). So that, the right side of (1. 1) is discrete form, i.e. a fuzzy set A can be determined by a sequence of ordinary sets $\{H(\lambda)\}$. Furthermore, we have proved that the membership function A(x) is a limit of a sequence of uniformly convergence functions.Of course, finding A(x) by using limit (or series) has the widespread using value.

It is called the discrete form of the decomposition theorem when a fuzzy set A is denoted by the union of the products of a sequence of sets and some numbers. Similarly, the discrete forms of the expression theorm and extension principle are also obtained.

§2. The discrete form of decomposition theorem

Given the domain X. Let A be a fuzzy set or a function from X into [0,1], and F(X), P(X) be fuzzy and ordinary power set on X, respectively. Let A(x) be the membership function of A. For $\lambda \in [0,1]$, set

$$A_\lambda=\{x\in X \mid A(x)\geq\lambda\}, \qquad A_{\underline{\lambda}}=\{x\in X \mid A(x)>\lambda\}$$

They are called λ—cut and strong λ—cut of A, respectively. The product λA of the number λ and A is the fuzzy set with the membership function $\lambda\wedge A(x)$.

We have

Theorem 2.1 Let D be a dense subset in [0,1]. Then,

$$A=\bigcup_{\lambda\in D} \lambda A_\lambda \qquad (2.1)$$

where $A\in F(X)$.

Proof. It is sufficent to prove that

$$A(x)=(\bigcup_{\lambda\in D}\lambda A_\lambda)(x), \quad \text{for } x\in X, \qquad (2.2)$$

In fact, for every $x\in X$, by the definition we obtain

$$(\bigcup_{\lambda\in D}\lambda A_\lambda)(x)=\bigvee_{\lambda\in D}(\lambda\wedge A_\lambda(x))=\bigvee_{\lambda\in D}\{\lambda \mid A(x)\geq\lambda\}=A(x)$$

So that, (2.2) holds. The proof is complete.

In theorem 2.1, the set D may be an arbitrary dense subset in [0,1] so that we can obtain a more general result. In some practice, it is possible that the less the elements of D, the better the result is. So, we can take D to be the dense countable subset in [0,1]—rational number set Q,and let $Q=\{r_1, r_2, \cdots, r_n, \cdots\}$.For a fuzzy set A,we have

$$A = \bigcup_{n=1}^{\infty} r_n A_{r_n}$$

That is A can be uniquely determined by a sequence of the ordinary sets $\{A_{r_n} \mid r_n \in Q, n=1,2,\cdots\}$. This decomposition of A is more simple that the expression discribed by an uncountable family of sets. From the point of view of applications, it is more conenient.

The same proof as theorem 2.1, we can show that the decomposition theorem denoted by strong cut is

Theorem 2.2. Let D be a dense subset in [0,1], then

$$A = \bigcup_{\lambda \in D} \lambda A_{\underset{\cdot}{\lambda}} \qquad (2.3)$$

where $A \in F(X)$.

A theorem that is more general than theorem 2.1,2.2 is

Theorem 2.3 Let D be a dense subset in [0,1], and suppose that the mapping

$$H: D \to P(X), \quad \lambda \to H(\lambda)$$

satisfies the relation

$$A_{\underset{\cdot}{\lambda}} \subset H(\lambda) \subset A_\lambda, \qquad \lambda \in D$$

Then,

1) $A = \bigcup_{\lambda \in D} \lambda H(\lambda)$ (2.4)

2) If $\lambda_1, \lambda_2 \in D$, and $\lambda_1 < \lambda_2$, then $H(\lambda_1) \supset H(\lambda_2)$

3) $A_\alpha = \bigcap_{\lambda \in D, \lambda < \alpha} H(\lambda) \quad (\alpha \in (0,1])$ (2.5)

$A_{\underset{\cdot}{\alpha}} = \bigcup_{\lambda \in D, \lambda > \alpha} H(\lambda) \quad (\alpha \in [0,1))$ (2.6)

Proof. 1) since $A_{\underset{\cdot}{\lambda}} \subset H(\lambda) \subset A_\lambda$, it follows that

$$\bigcup_{\lambda \in D} \lambda A_{\underset{\cdot}{\lambda}} \subset \bigcup_{\lambda \in D} \lambda H(\lambda) \subset \bigcup_{\lambda \in D} \lambda A_\lambda$$

By (2.1) and (2.3), we obtain the formula (2.4).

2) Since $\lambda_1 < \lambda_2$, so $H(\lambda_1) \supset A_{\underset{\cdot}{\lambda_1}} \supset A_{\lambda_2} \supset H(\lambda_2)$

3) For every fixed $\alpha \in (0,1]$, we obtain

$$\bigcap_{\lambda \in D, \lambda < \alpha} H(\lambda) \subset \bigcap_{\lambda \in D, \lambda < \alpha} A_\lambda = A_{\left(\bigvee_{\lambda \in D, \lambda < \alpha} \lambda\right)} = A_\alpha$$

and when $\lambda < \alpha, \lambda \in D$, the relation $A_\alpha \subset A_{\underset{\cdot}{\lambda}} \subset H(\lambda)$ holds, so that

$$A_\alpha \subset \bigcap_{\lambda \in D, \lambda < \alpha} H(\lambda)$$

and it follows that (2.5) holds.

A similar argument shows that (2.6) holds,and the proof is complete.

Since $H(\lambda)$ may be not the cut or strong cut of A, so that the formula (2.4) is an extension of (2.1) and (2.3). For (2.4), it is noted that the range D of λ is more "smaller" that the interval [0,1], then it is useful from the point of view of applications.

If D is a dense countable subset in [0,1] and changing the formula (2.1), (2.3) and (2.4) into the express forms of functions, then the membership function of $A(x)$ is the limit of a sequence of functions or function series. In the following, we shall prove this result by taking theorem 2.1 as an example.

Theorem 2. 4 Let D be a dense countable subset in $[0,1]$ and $D=\{\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots\}$ Then, the sequence of sets

$$A^{(n)} = \bigcup_{i=1}^{n} \lambda_i A_{\lambda_i}, \qquad n=1,2,\cdots$$

is a monotone sequence, and $\{A^{(n)}(x)\}$ converges uniformly to $A(x)$ with respect to x in X.

Proof. since

$$A^{(n)}(x) \le [A^{(n)}(x)] \vee [\lambda_{n+1} \wedge A_{\lambda_{n+1}}(x)] = A^{(n+1)}(x), \quad n=1,2,\cdots$$

it follows that $A^{(n)} \subset A^{(n+1)}$ for $n=1,2,\cdots$.

We shall show that $\lim A^{(n)}(x)=A(x)$ uniformly with respect to x in X. For arbitrary $\varepsilon>0$, first, we take a natural number m and $\frac{1}{m}<\frac{\varepsilon}{2}$, divide the interval $[0, 1]$ equally into m-parts. By the dence of D in $[0,1]$, we can take λ_{k_i} in $\left(\frac{i-1}{m}, \frac{i}{m}\right)$ and $\lambda_{k_i} \in D$, $(i=1,2,\cdots,m)$. Then

$$\lambda_{k_1} < \frac{\varepsilon}{2}, \quad \lambda_{k_m}+\frac{\varepsilon}{2}>1, \quad 0<\lambda_{k_i}-\lambda_{k_{i-1}}<\varepsilon, \text{ for } i=2,\cdots,m.$$

Again, take $N=\max\{k_1, k_2, \cdots, k_m\}$, and the first N-elements $\lambda_1, \cdots, \lambda_N$ in D are rearranged from small to large, say $\lambda_1^* < \lambda_2^* < \cdots < \lambda_N^*$

If necessary, we may set $\lambda_0^*=0$ and $\lambda_{N+1}^*=1$. It follows that

$$\max_{0\le i\le N} (\lambda_{i+1}^* - \lambda_i^*) < \varepsilon$$

Since $0=\lambda_0^* \le \lambda_1^* < \lambda_2^* < \cdots < \lambda_N^* \le \lambda_{N+1}^*=1$, then for every $x \in X$, When $A(x)<1$, there exists an integer i $(0\le i\le N)$ such that

$$\lambda_i^* \le A(x) < \lambda_{i+1}^*$$

$$A^{(N)}(x) = \bigvee_{k=1}^{N} (\lambda_k \wedge A_{\lambda_k}(x) = \bigvee_{k=1}^{N} (\lambda_k^* \wedge A_{\lambda_k^*}(x)) = \lambda_i^*$$

So it can be seen that

$$0 \le A(x)-A^{(N)}(x) = A(x)-\lambda_i^* \le \lambda_{i+1}^* - \lambda_i^* < \varepsilon$$

When $A(x)=1$, then $A^{(N)}(x)=\lambda_N^*$, Also, it follows that

$$0 \le A(x)-A^{(N)}(x) = 1-\lambda_N^* < \varepsilon$$

As above, for arbitrary $\varepsilon>0$, there exists a positive integer N such that when $n\ge N$,

$$0<A(x)-A^{(n)}(x)<A(x)-A^{(k)}(x)<\varepsilon, \text{ for all } x\in X$$

i. e., $A^{(n)}(x)$ converges uniformly to $A(x)$ with respect to x in X. the proof is complete.

By the result of theorem 2.4, it can be seen that A is not only determined uniquely by a sequence of the ordinary sets, but also it is denoted by the limit of a monotone seqence of sets $\{A^{(n)}\}$, and the sequence of the functions $\{A^{(n)}(x)\}$ corresponding to $\{A^{(n)}\}$ converges uniformly to $A(x)$ with respective to $x\in X$. For fixed n, the range of $A^{(n)}(x)$ is only the finite subset of D. This sets the fundamentation for applications and approximate evaluations.

§3 The discrete form of expression theorem

Let D be a dense countable subset in $[0,1]$ and if $\lambda\in D$, then $1-\lambda\in D$. setting

$$D=\{\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots\}$$

Definition 3.1 If the mapping

$$H: D\to P(X), \ \lambda_i \to H(\lambda_i)$$

satisfies

$$H(\lambda_i)\supset H(\lambda_j), \text{ for } \lambda_i<\lambda_j \text{ and } \lambda_i, \lambda_j\in D$$

Then the mapping H is called a sequence of sets preserving reverse order with respect to D on X. The set of all H with the properties of definition 3.1 is denoted by $\mathcal{U}_D(X)$.

Definition 3.2 For the elements in $\mathcal{U}_D(X)$, the operators $\cup, \cap$ and C are definited as follows,respectively.

$$\bigcup_{\gamma\in\Gamma} H_\gamma: \quad (\bigcup_{\gamma\in\Gamma} H_\gamma)(\lambda_i)=\bigcup_{\gamma\in\Gamma} H_\gamma(\lambda_i)$$

$$\bigcap_{\gamma\in\Gamma} H_\gamma: \quad (\bigcap_{\gamma\in\Gamma} H_\gamma)(\lambda_i)=\bigcap_{\gamma\in\Gamma} H_\gamma(\lambda_i)$$

$$H^c: \quad H^c(\lambda_i)=[H(1-\lambda_i)]$$

where $\lambda_i\in D$, Γ is an index set.

Theorem 3.1 Let the mapping

$$T: \mathcal{U}_D(X)\to F(X), \quad H\to T(H)=\bigcup_{i=1}^{\infty}\lambda_i H(\lambda_i)$$

Then, T is a surjective homomorphism from $(\mathcal{U}_D(X), \cup, \cap, C)$ onto $(F(X), \cup, \cap, C)$, and

$$T(H)_{\underset{\cdot}{\lambda_i}}\subset H(\lambda_i)\subset T(H)_{\lambda_i}, \quad (\lambda_i\in D), \qquad (3.1)$$

$$T(H)_\alpha=\bigcap_{\lambda_i<\alpha} H(\lambda_i), \quad (\alpha\in(0,1]), \qquad (3.2)$$

$$T(H)_\alpha = \bigcup_{\lambda_i > \alpha} H(\lambda_i), \quad (\alpha \in [0,1)), \qquad (3.3)$$

Proof. 1) For $H \in \mathcal{U}_D(X)$, Since

$$T(H) = \bigcup_{i=1}^{\infty} \lambda_i H(\lambda_i)$$

Hence $T(H)$ is a defined-element in $F(X)$ and T is a mapping from $\mathcal{U}_D(X)$ to $F(X)$.

2) For $A \in F(X)$, setting $H(\lambda_i) = A_{\lambda_i}$, $(\lambda_i \in D)$ then $H \in \mathcal{U}_D(X)$ and by theorem 2.1, we get $T(H) = A$. i.e. T is a surjective mapping from $\mathcal{U}_D(X)$ onto $F(X)$.

3) First, $x \notin H(\lambda_i) \Rightarrow H(\lambda_i)(x) = 0$

$\Rightarrow \forall \alpha > \lambda_i,\ \alpha \in D,\ H(\alpha)(x) = 0$

$$\Rightarrow T(H)(x) = \bigvee_{\alpha \in D} (\alpha \wedge H(\alpha)(x)) = \bigvee_{\alpha \in D \cap [0,\lambda_i)} (\alpha \wedge H(\alpha)(x)) < \bigvee_{\alpha \in [0,\lambda_i)} \alpha = \lambda_i$$

$\Rightarrow x \notin T(H)_{\lambda_i}$

Hence $T(H)_{\lambda_i} \subset H(\lambda_i)$.

Second, $x \in H(\lambda_i) \Rightarrow H(\lambda_i)(x) = 1$

$$\Rightarrow T(H)(x) = \bigvee_{\alpha \in D} (\alpha \wedge H(\alpha)(x)) \geq \lambda_i \wedge H(\lambda_i)(x) = \lambda_i \Rightarrow x \in T(H)_{\lambda_i}$$

It follows that $H(\lambda_i) \subset T(H)_{\lambda_i}$ and (3.1) holds.

4) From (3.1) and theorem 2.3, it follows that (3.2) and (3.3) hold.

5) By (3.3), $\forall \alpha \in [0,1)$, we obtain

$$T(\bigcup_{\gamma \in \Gamma} H_\gamma)_\alpha = \bigcup_{\lambda_i > \alpha} (\bigcup_{\gamma \in \Gamma} H_\gamma)(\lambda_i) = \bigcup_{\lambda_i > \alpha} (\bigcup_{\gamma \in \Gamma} H_\gamma(\lambda_i))$$

$$= \bigcup_{\gamma \in \Gamma} (\bigcup_{\lambda_i > \alpha} H_\gamma(\lambda_i)) = \bigcup_{\gamma \in \Gamma} T(H_\gamma)_\alpha = (\bigcup_{\gamma \in \Gamma} T(H_\gamma))_\alpha$$

From theorem 2.2, it follows that

$$T(\bigcup_{\gamma \in \Gamma} H_\gamma) = \bigcup_{\gamma \in \Gamma} T(H_\gamma)$$

6) From (3.2), we get

$$T(\bigcap_{\gamma \in \Gamma} H_\gamma)_\alpha = \bigcap_{\lambda_i < \alpha} (\bigcap_{\gamma \in \Gamma} H_\gamma)(\lambda_i) = \bigcap_{\lambda_i < \alpha} (\bigcap_{\gamma \in \Gamma} H_\gamma(\lambda_i))$$

$$= \bigcap_{\gamma \in \Gamma} (\bigcap_{\lambda_i < \alpha} H_\gamma(\lambda_i)) = \bigcap_{\gamma \in \Gamma} T(H_\gamma)_\alpha = (\bigcap_{\gamma \in \Gamma} T(H_\gamma))_\alpha$$

By theorem 2.1, it follows that

$$T(\bigcap_{\gamma \in \Gamma} H_\gamma) = \bigcap_{\gamma \in \Gamma} T(H_\gamma)$$

7) For fixed $\alpha \in (0,1]$

$$T(H^c)_\alpha = \bigcap_{\lambda_i < \alpha} H^c(\lambda_i) = \bigcap_{\lambda_i < \alpha} (H(1-\lambda_i))^c$$

$$= (\bigcup_{1-\lambda_i > 1-\alpha} H(1-\lambda_i))^c = (T(H)_{(1-\alpha)})^c = ((T(H))^c)_\alpha$$

From theorem 2.1, we have

$$T(H^c)=(T(H))^c$$

From the above 1), 2), 5), 6) and 7), we have showed that T keeps union, meet and comlementation operators and T is a surjective homomorphism. The proof is complete.

In general, we also point out that T is not an injection. For example, let $A\in F(X)$, $H_1(\lambda_i)=A_{\underset{\cdot}{\lambda_i}}$, $H_2(\lambda_i)=A_{\lambda_i}$ $(\lambda_i\in D)$ and when $\{A_{\underset{\cdot}{\lambda_i}}\mid \lambda_i\in D\}\neq\{A_{\lambda_i}\mid \lambda_i\in D\}$, it follows that $H_1\neq H_2$ and $T(H_1)=T(H_2)$ by theorem 2.3.

If we define a relation "$\backsim$" on $\mathcal{U}_D(X)$ as follows

$$H'\backsim H \Leftrightarrow T(H')=T(H)$$

It is easily checked that "$\backsim$" is an equivalent relation. By the relation "$\backsim$", we can compose the quotient set $\mathcal{U}_D(X)/\backsim$ which is an isomorphic with F(X) as long as defining a proper operator.

§4. The discrete form of extension principle

Let X and Y be two nonempty sets. The mapping

$$f:\ X\to Y,\qquad x\to f(x)$$

The Zadeh's extension principle is

Definition 4.1 Let the mapping f be from X into Y.we can get two induced mappings f that is from F(X) into F(Y) and f^{-1} that is from F(Y) into F(X) as follows

$$f:\ F(X)\to F(Y),\quad A\to f(A)$$

$$f^{-1}:\ F(Y)\to F(X),\quad B\to f^{-1}(B).$$

where the definitions of their membership functions are

$$f(A)(y)=\bigvee_{f(x)=y}A(x)\quad\text{and}\quad f^{-1}(B)(x)=B(f(x)),$$

respectively.

An equivalent result to the definition 4.1 is

Theorem 4. 1 Let the mapping f from X into Y and $A\in F(X)$, $B\in F(Y)$. If the fuzzy sets f(A) and $f^{-1}(B)$ are defined as above. Then

$$f(A)=\bigcup_{\lambda\in[0,1]}\lambda f(A_\lambda)=\bigcup_{\lambda\in[0,1]}\lambda f(A_{\underset{\cdot}{\lambda}})$$

$$f^{-1}(B)=\bigcup_{\lambda\in[0,1]}\lambda f^{-1}(B_\lambda)=\bigcup_{\lambda\in[0,1]}\lambda f^{-1}(B_{\underset{\cdot}{\lambda}})$$

In theorem 4.1, the results hold when the index set [0,1] is replaced by its dence subset.

Theorem 4.2 Let D be a dence subset in [0,1] and the mapping $f:\ X\to Y$. then

$$f(A)=\bigcup_{\lambda\in D}\lambda f(A_\lambda),\quad \text{for } A\in F(X)$$

$$f^{-1}(B)=\bigcup_{\lambda\in D}\lambda f^{-1}(B_\lambda),\quad \text{for } B\in F(Y)$$

The proof is simple.

Theorem 4.3 Let D be a dence subset in [0,1], the mapping f is from X into Y, $A\in F(X)$, and $B\in F(Y)$. Then

1) $f(A)=\bigcup_{\lambda\in D}\lambda f(A_{\underset{\cdot}{\lambda}})$

2) $f^{-1}(B)=\bigcup_{\lambda\in D}\lambda f^{-1}(B_{\underset{\cdot}{\lambda}})$

3) If the mapping $H: D\to P(X)$, $\lambda\to H(\lambda)$ satisfies $A_{\underset{\cdot}{\lambda}}\subset H(\lambda)\subset A_\lambda$, $\lambda\in D$

Then $f(A)=\bigcup_{\lambda\in D}\lambda f(H(\lambda))$

4) If the mapping $H': D\to P(Y)$, $\lambda\to H'(\lambda)$ satisfies $B_{\underset{\cdot}{\lambda}}\subset H'(\lambda)\subset B_\lambda$, $\lambda\in D$

Then $f^{-1}(B)=\bigcup_{\lambda\in D}\lambda f^{-1}(H'(\lambda))$.

The proof is omitted.

We can chose D that is a countable dence subset in [0,1] and $D=\{\lambda_1,\lambda_2,\cdots,\lambda_i,\cdots\}$. Give the mapping $f: X\to Y$ and $A\in F(X)$, then

$$f(A)=\bigcup_{i=1}^{\infty}\lambda_i f(A_{\lambda_i})$$

We can use the above formula to find out f(A) or caluculate approximatly $f(A)(y)$ $(y\in Y)$. The result may be helpful to solve the practical problems.

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